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A

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CONTAINING

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INCLUDED IN THE REGULATIONS OF THE SENATE,
FOR THE MATRICULATION AND BACHELOR OF ARTS AND
BACHELOR OF SCIENCE PASS EXAMINATIONS;

WITH

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THOMAS KIMBER, M.A. LOND.

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TO

THE REV. R. W. JELF, D.D.,

PRINCIPAL OF KING'S COLLEGE,

LONDON:

AS A TRIBUTE

OF SINCERE ESTEEM FOR HIS TALENTS AND LEARNING,

OF ARDENT ADMIRATION OF HIS

CHRISTIAN PRINCIPLES

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⁂ At the suggestion of several eminent teachers this part of the Mathematical Course has been published separately for the convenience of Schools preparing pupils for the Matriculation Examination. It includes all the papers connected with that Examination since 1838, and as they comprise upwards of a thousand questions in the leading rules of Algebra and Arithmetic, they will, it is hoped, be found to furnish a sufficient variety for practice.

To render it more suitable for school purposes, the solutions have been removed altogether from the questions, and are printed separately.

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INTRODUCTION.

◆

DEFINITIONS,

with an explanation of the principal symbols of arithmetic and algebra.

1. FOREIGNERS, ignorant of each other's language, make use of their fingers to express numbers. The same is generally done in conversing with the dumb, and in almost every case where written or vocal symbols cannot be employed. To this practice may be traced the origin of all systems of numerical notation. It has been so invariable as to seem quite a *natural* provision rather than an artificial invention; and hence, in our own scheme, the signs of which it is composed are called *natural* numbers, and from their evident connexion with the use of the *fingers* they are also called *digits*. If man had been constituted a twelve-fingered animal, we should now have possessed a much more convenient system of numeration than we do. From unity up to *ten*, the signs and corresponding names vary with every different number. In writing ten and higher numbers, to avoid the obvious inconvenience of continuing to invent new signs and names, a cipher is introduced, which is not itself significant, but employed merely to point out the position, or give a *local value* to the significant figures to which it is annexed. Neither the Greek nor Roman scales of notation possessed the advantages of local value; this principle was derived from the Hindoos, and first introduced into Europe about the eleventh century.

2. *Notation* is the art of expressing any number by figures, which is already given in words.

Numeration is the converse of Notation, being the art of expressing any number in words which is already given in figures.

3. *The Decimal Notation.*—The *local value* of a figure is known by its position with respect to other numbers or ciphers to which it is joined. In the first or units' place, all figures have their initial value; in the second, reckoning to the left, or tens' column, the same figures represent ten times their initial value; and in the third or hundreds' column, one hundred times their initial value, and so on as represented in the following table.

Numeration Table.

&c.	Tens of Millions of Millions.	Millions of Millions.	Hundrds. of Thds. of Millns.	Tens of Thousands of Millns.	Thousands of Millions.	Hundreds of Millions.	Tens of Millions.	Millions.	Hundreds of Thousands.	Tens of Thousands.	Thousands.	Hundreds.	Tens.	Units.
7	9	2	8	4	6	5	3	1	7	4	8	6	0	2

☞ In the above number, as in all integer numbers, the digit on the extreme right expresses units; there are no tens in the number, therefore a cipher is put in the second place, otherwise the remaining digits on the left would not retain their proper value, the hundreds would become tens, the thousands hundreds, and so on. The third digit 6 expresses its units of hundreds, and the fourth 8 its units of thousands, and in the same way each of the remaining digits has its *local* as well as its *initial* signification. After the millions' column, most arithmetical writers employ the words billion, trillion, quadrillion, quintillion, sextillion, septillion, octillion, nonillion, &c., but these terms are never in common use, the want of such high numbers being seldom felt. The notation adopted in the numeration table above, is much simpler. No denomination higher than a million is required, all the figures, in any very high number, after six are counted off towards the left, as so many millions, and after twelve figures are thus counted off, so many millions of millions.

4. When numbers are used to indicate the *kind* or *denomination* of the units employed, as 3 *pence*, 5 *yards*, 7 *stones*, they are all called *concrete;* when used without reference to any such denomination, and simply to express a certain *number*, they are called *abstract*.

5. The word calculation is derived from the Latin word *calculus*, a pebble; and as all the processes in arithmetic consist of the increase and diminution of numbers, most of these were originally carried on by the use of pebbles, and all may still be so transacted.

The four fundamental rules of arithmetic are, Addition, Subtraction, Multiplication, and Division. The last two, however, are only compendious methods of doing the first and second, as we shall hereafter show.

6. If we wish to reason about numbers generally, without confining ourselves to any case in particular, we make use of letters. Thus let f stand for a number, and x, y, z, for the parts into which it is divided, then whatever results we obtain from these letters, are not results that belong to any particular set of numbers, but such as are equally true of all. This science of general reasoning on

the properties of numbers is called ALGEBRA. To avoid repetition, we shall treat of arithmetic and algebra under the same heads, since the rules for processes in both are identical.

Known numbers or magnitudes are usually denoted by the first letters of the alphabet, as A, B, C, D, and a, b, c, d, and α, β, γ, δ, &c.; and unknown quantities by other letters, as v, w, x, y, z, or θ, ϕ, ψ, &c.

The principal signs, requiring explanation, are the following: $+ \ - \ \times \ \div \ = \ > \ < \ \therefore \ \because$

$+$, which is read *plus*, shows that the quantity before which it stands is to be added.

$-$, which is read *minus*, shows that the quantity before which it stands is to be subtracted.

$\times$, which is read *into*, shows that the quantities between which it stands are to be multiplied. Sometimes a point is used instead of this sign thus, $5 \,.\, 6$ instead of 5×6; and between letters no sign is used, $a\,x$ means $a \times x$.

$\div$, which is read *by*, shows that the quantity which stands before it is to be divided by the one which follows; a better way is to represent the two numbers as a fraction thus, $\frac{a}{b}$, instead of $a \div b$, or $a : b$.

$=$, which is read *equals* or *equal*, shows that the quantities between which it stands are equal; between the terms of a proportion this sign $::$ is used.

$>$ and $<$ are signs of *inequality*, which show that one quantity is greater or less than another; as, $a > b$, which is read a is greater than b; and $a < b$, which is read a is less than b.

$\therefore$, read *therefore*, or *then*, or *consequently*.

$\because$, read *since or because*.

The vinculum ——, placed over, and the brackets ( ), including two or more algebraic terms, show that these terms are to be considered as one quantity.

7. The continual recurrence of the same quantities renders abbreviation necessary; thus when the sum of three numbers, each equal to x, is required, $3\,x$ is written instead of $x + x + x$. The number 3 is called the *coefficient* of x, and where letters are used as in $a\,x$, and $3\,b\,x$, a and $3\,b$ are the coefficients of x. When unity is the coefficient of an algebraical quantity, it is never expressed.

The parts, or individual letters or numbers, of which a product is composed are called *factors;* thus a, or x, is a *factor* of ax, and 3, b, and x, are respectively *factors* of $3\,bx$.

When a quantity is multiplied by itself, an abbreviation is employed thus:—

$x\,x$ is called the second power of x, and is written x^2
$x\,x\,x$ „ third „ „ x^3
$x\,x\,x\,x$ „ fourth „ „ x^4

and the same for higher numbers. Here 2, 3, and 4, are called the *exponents* or *indices* of x, and are used to point out the power to which x is raised.

Quantities which contain the same letter, or the same combination of letters, are said to be *like*, and those which are composed of different letters, or different combinations of letters, are called *unlike*. Thus $3\,x$, $4\,x$; $5\,x\,y$, $7\,x\,y$; $8\,x\,y\,z$, $9\,x\,y\,z$, are pairs of *like* quantities; while $3\,x$, $4\,y$, $5\,x^2$, $7\,x\,z$, are *unlike* quantities.

The sign $\sqrt{\ }$, an extension of the letter r, the first letter of the word *radix*, is placed over a quantity when its root is to be extracted, the particular root being shown by a small figure prefixed to the sign, thus $\sqrt[3]{x}$, is the cube or third root of x; the square or second root of a quantity is written thus, $\sqrt{x}$, not $\sqrt[2]{x}$. In some cases these fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., are used instead of these symbols, as explained in a succeeding chapter (Art. 82); thus, $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{1}{4}}$, &c., mean the square, cube, fourth, &c., root of x.

8. The distinction between arithmetic and algebra, and some of the advantages of the latter, may be made clearer by an example.

Ex. The sum of two numbers is 98, and their difference 24, find them.

Let L represent the less of the two, then L added to the difference will represent the greater; and the less added to the greater gives 98. Or, in an algebraic form we have the following equations:—

$$\text{The greater is } L + 24, \text{ the less is } L;$$

$$\therefore 2\,L + 24 = 98.$$

Subtract 24 from both sides, which does not destroy the equality,

$$\text{Then } 2\,L = 74, \therefore L = 37,$$

$$\text{And the greater is } 61.$$

The preceding formula not only solves the particular question asked, but will suit *any* two numbers whatever whose sum and difference are given.

9. Questions in Algebra, like the propositions in Euclid, are of two kinds,—problems and theorems.

In a problem, some number or quantity has to be found, of a certain value, according to the terms of the question. The preceding question is a problem. In a theorem, some given proposition has to be proved to be true in all cases. The following is a theorem: If a and b be any two numbers of which a is the greater, then

$$a = \frac{a+b}{2} + \frac{a-b}{2}.$$

CHAPTER I.

THE ORDINARY RULES OF ARITHMETIC AND ALGEBRA.

ADDITION AND SUBTRACTION,

with an explanation of the use of brackets, and of the signs plus and minus.

10. In adding one number to another, we signify the process by the sign +, thus 6 + 7 means, that 6 is to be added to 7, and the result 13, is called their *sum*.

To add integer numbers together, place them with their units', tens', &c., figures in vertical columns, and draw a straight line below the lowest number; begin by adding the units' column, then the remaining columns, as shown below for annexed example:

$$\begin{array}{r}376\\5241\\78432\\19\\\hline 84068\end{array}$$

9 + 2 + 1 + 6 = 18 units, which represented more simply are 1 ten + 8 units, or 10 + 8; place each of these numbers in their proper columns beneath the line, *i.e.*, *write down* 8 in the units', and *carry* 1 to the tens' column, and proceed again, thus:—

1 to carry + 1 + 3 + 4 + 7 = 16 tens, which are 1 hundred and 6 tens, or 100 + 60; write down 6 in the tens' and carry 1 to the hundreds' column, and as before say,—

1 to carry + 4 + 2 + 3 = 10 hundreds or 1000; now place a cipher in the hundreds', carry 1 to the thousands' column, and say—

1 to carry + 8 + 5 = 14 thousands, which are 10,000 + 4000; write down 4 in the thousands', and carry 1 to the ten thousands' column; then,

1 to carry + 7 = 8 ten thousands, or 80,000; place 8 in the ten thousands' column, and you obtain the *sum* which was required.

11. In Subtraction one number is taken from another, and the process is expressed by the sign —. Thus, 9 — 2, means that 2 is to be subtracted from 9, and the result, 7, is the *difference* or *remainder*.

To subtract one integer number from another, place the less, called the *subtrahend*, under the greater, with their units', tens', &c., figures in vertical columns, and draw a straight line below the subtrahend; begin at the units'

$$\begin{array}{r}68472\\24541\\\hline 43931\end{array}$$

column by subtracting the lower digit from the upper, and proceed to do the same with the remaining columns, as shown below for annexed example:

$2 - 1 = 1$, write down 1 in the units', and then proceed to the tens' column; here

$7 - 4 = 3$ tens, or 30, write down 3 in the tens', and proceed to the hundreds' column; next say

$4 - 5$ you cannot, but borrow 10 (units of the hundreds', or 1 of the thousands' column, from the 8000, leaving 7000); then

$14 - 5 = 9$ hundreds, or 900, place 9 in the hundreds', and proceed to the thousands' column.

(Here you may take 4 from 7, as the French and Germans do, but it is more convenient, we think, and therefore our practice, to leave the upper figure unaltered and to carry 1 to the lower, the process then is)

1 to carry $+ 4 = 5$, and $8 - 5 = 3$ thousands, or 3000; place 3 in the thousands', and proceed to the ten thousands' column.

$6 - 2 = 4$ ten thousands, or 40,000; place 4 in the ten thousands' column, and the subtraction is completed.

NOTE.—The principle upon which the process of borrowing ten, as in the preceding case, depends, is the following:—If there are two heaps of pebbles, one consisting of 8000 and the other of 4000, the difference is 4000, and if 1000 more be added to each heap, the difference will still be 4000; hence, in subtraction, we may for convenience add any number whatever to both terms, without altering the remainder.

12. In algebra, the expression $a + b$ signifies the sum of the two numbers represented by a and b; and $a - b$ signifies that the number to be represented by b is to be subtracted from the number represented by a.

The reduction of several terms into one can be performed when there are several terms which are similar as to letters, and alike or unlike as to signs; thus $6a + 3a = 9a$, and $- 3a - 2a = - 5a$, also $a + 12a - 3a - 6a + 2a - a = 5a$. Here all the terms to be added amount to $15a$, and those to be subtracted to $10a$; that is, $10a$ is to be subtracted from $15a$, and the result is $5a$. Similarly $a + b - 3a + 4b = 5b - 2a$; because b and $4b$ added together give $5b$, and adding a and then subtracting $3a$, is the same thing as subtracting $2a$.

It is quite immaterial in what order the numbers are written, provided the proper sign is prefixed to each. For instance $12 + 2 - 5 - 3 - 1$, or $2 - 1 - 3 - 5 + 12$, or $2 + 12 - 1 - 5 - 3$. Nor will any greater difficulty be experienced in determining algebraic results, $a - b - c + d - e$, where a, b, c, d, e, are used to represent numbers. For it is evident, that from the numbers represented by a and d, to which $+$ is prefixed, are to be subtracted those represented by the letters b, c, e, which are preceded by $-$.

The rule for addition of algebraic quantities, as may be inferred from the preceding article is—

Consider all the terms collectively as forming one expression, and make all possible reductions which similarity of terms, with like or unlike signs, will allow.

13. As a common source of error in practice is a mistake in the signs, too much attention can scarcely be given to ascertain their meaning and the rules for using them. Those numbers which have the sign + prefixed are called *positive*, those with the sign —, *negative* quantities.

14. These signs, when joined to numbers, represent addition and subtraction; but they are not limited to this meaning, and in algebra have a much more extensive signification. Whatever quantity $+ a$ represents, $- a$ represents the contrary. Thus: if $+ a$ represents property, $- a$ represents debt; if $+ a$ be the length of a right line or curve in any direction, $- a$ is the same length of a right line or curve in the opposite direction.

15. Any expression between brackets must be treated as a single quantity. If a, b, and c be three numbers, of which a is greater than b, and b greater than c, then

$$(a + c) + (b - c) = a + b$$
$$\text{and } a + c + b - c = a + b$$

are the same; for both equations are only the algebraical form of saying that the sum $a + b$ continues unaltered, if you add c to one and subtract it from the other. From this conclusion, and from similar results in like cases, we obtain the rule for the removal of brackets when preceded by a positive sign. *When the bracket is preceded by a positive sign, it may be omitted without altering the value of the quantities it includes.*

16. To show how to remove the brackets from an expression when a negative sign precedes them, we will take the numbers $17 - (5 + 2)$. If we remove the bracket and subtract 5 from 17 we subtract too little, as $5 + 2$ were to be subtracted, and we have therefore taken too little by 2. The result, however, will be correct if we take both 5 and 2 from 17; and to represent this when the brackets are removed, we must write the expression, $17 - 5 - 2$.

Now to remove the brackets from the expression $17 - (5 - 2)$, if we take away the brackets and subtract 5 we subtract too much, as it is not 5 that is to be subtracted, but 5 diminished by 2; we have therefore taken away 2 too much, but the result will be correct if we add 2; and the expression will then be, after removing the brackets, $17 - 5 + 2$.

17. Similarly in an algebraic expression, *e.g.*, $a - (b + c)$. Take away the brackets, and subtract b from a; too little is sub-

tracted: because from a, both b and c should have been taken. Therefore too little by c has been taken, and therefore c also must be taken; and the expression without the brackets should be $a - b - c$.

Again: in the expression $a - (b - c)$, if from a we take b, we subtract too much; because only b diminished by c should have been subtracted, and c too much has therefore been taken away. If however we now add c, the result will be correct; and the expression without the brackets should be, $a - b + c$.

$$\left.\begin{aligned} \therefore\ a - (b + c) &= a - b - c. \\ \text{and } a - (b - c) &= a - b + c. \end{aligned}\right\} (A.)$$

As an extension of the above, take the following example :—

$$a - (b + c - m - n + r). \qquad (B.)$$

If two quantities be increased equally, their difference continues unaffected, (see Art 11, Note,) therefore add $(m + n - r)$ to each of the above, then they become

$$a + m + n - r - (b + c - m - n + r + m + n - r),$$

which are the same as $a + m + n - r - (b + c)$,
which by $(A.)$ is the same as $a + m + n - r - b - c$,

$$\text{or, } a - b - c + m + n - r. \qquad (C.)$$

On comparing B. and C. together, the following rule of signs is deduced:

When the bracket is preceded by a negative sign, it may be omitted, if all the signs it includes are changed.

The general rule for the subtraction of all algebraic quantities is also obtained from the same chain of reasoning.

Change the signs of the quantities that are to be subtracted, and then proceed as in addition.

18. When a negative quantity precedes a bracket within which is another bracket, and within that a third, and so on, the quantities which are an even number of times under the negative sign are positive, those preceded by an odd number of negative signs remain negative, as the following example will show:

$$a - \{b - [c - (m - n)]\} = a - b + c - m + n.$$

It is immaterial, in removing several brackets from an expression, whether the innermost or the outermost bracket be first removed. The rule of signs should be so familiar to the student as to enable him to remove the brackets from an expression at one operation, as in the example above.

EXAMPLES.

$$\text{(1) Add } \begin{array}{r} 1 + 2x - 4y \\ \text{To } 2 - 3x + 6y \\ \hline 3 - x + 2y \end{array} \qquad \text{(2) } \begin{array}{r} 7ax + 6by - 3cz \\ 6ax - 3by + 2cz \\ \hline 13ax + 3by - cz \end{array}$$

(3)
$$\begin{array}{r}
3x^2 - 2xy \\
4x^2 + 6xy \\
-x^2 + xy \\
\hline
6x^2 + 5xy
\end{array}$$

(4)
$$\begin{array}{r}
3a + 4b + 5c \\
2a - 9b + 6c \\
4a - 3b - 7c \\
\hline
9a - 8b + 4c
\end{array}$$

(5)
$$\begin{array}{lr}
\text{From} & 5x^2 + 6xy + y^2 \\
\text{Take} & 4x^2 - 3xy + 2y^2 \\
\hline
\text{Diff.} & x^2 + 9xy - y^2
\end{array}$$

(6)
$$\begin{array}{r}
2a - 3b - a + b \\
2a - 3b - a - b \\
\hline
* \quad * \quad * \quad 2b
\end{array}$$

(7)
$$\begin{array}{r}
7x^2 - 8xy + 9y^2 \\
5x^2 + 11xy + 8y^2 \\
\hline
2x^2 - 19xy + y^2
\end{array}$$

MULTIPLICATION.

19. In addition and subtraction no abstract numbers are necessary; we might perform every operation in either rule on the quantities themselves. But in multiplication and division, where one set of numbers has to be taken a number of *times*, we unavoidably use abstract numbers.

20. Multiplication may be considered as a short method of performing addition: thus $a + a + a$ is the same as $a \times 3$, which is written $3a$. Here a is the *multiplicand*, 3 is the *multiplier*, and the result obtained, $3a$, is the *product*.

21. The several steps taken in multiplying in the ordinary way by a single number, are as follows:

To multiply 1756 by 4. Referring to the principle of local value explained in Art. 3,

$$1756 = 1000 + 700 + 50 + 6,$$

and to multiply the whole number by 4, we must multiply each of those parts by 4.

$$\begin{array}{rcrcr}
6 & \times & 4 & = & 24 \\
50 & \times & 4 & = & 200 \\
700 & \times & 4 & = & 2800 \\
1000 & \times & 4 & = & 4000 \\
\hline
1756 & \times & 4 & = & 7024
\end{array}$$

It is quite apparent that if we *carry*, as was done in addition, from the column we are multiplying, the units belonging to the next higher column, only one line will be necessary. Thus, multiplying the units' column, we obtain 24; the 2 belongs to the tens' column, and must be added to the product obtained by the next step. Thus 2 to carry + 4 × 5 = 22; place 2 in the tens' and carry 2 to the hundreds' column. Proceed in the same way with the remaining figures, and the multiplication is completed.

22. Suppose it is required to multiply 1756 by 304. Here $304 = 300 + 4$. Each digit of the multiplicand is multiplied separately by each digit of the multiplier, and the products are arranged in separate lines: the meaning of each line is indicated on the left of the process.

$$\begin{array}{rrr} & & 1756\\ & & 304\\ \cline{3-3} 1756 \times & 4 = & 7024\\ 1756 \times & 300 = & 5268\\ \cline{3-3} & & 533824 \end{array}$$

In this operation, after multiplying by 4, the 0 in the multiplier is passed over; and proceeding to multiply by 3, the figure obtained from the units' column is placed in the hundreds' column below 0, and each succeeding figure one place to the left, as before. In the same way any higher numbers may be multiplied together.

23. In the multiplication of algebraic quantities, the product of two positive quantities, it is impossible to doubt, is positive. The product of $-a$ and $+3$, means $-a$ taken three times, which evidently is $-3a$. In like manner any other quantities, as $-a$ taken b times give $-ab$. Hence we obtain the rule:—*A positive quantity multiplied by a negative one, or a negative quantity by a positive, produces a negative result.*

24. In multiplying two negative quantities, as $-a$ by $-b$, the result must be either positive or negative, and it cannot be negative because it must produce a contrary result to $-a$ multiplied by $+b$, therefore it must be positive.

25. The rules of signs, therefore, to be attended to in multiplication, are:

Like signs multiplied together give $+$.
Unlike signs . . . . $-$.

26. In compound multiplication, where it is requisite to multiply any quantity, simple or compound, by a compound quantity, *e.g.* $a + b$ by $a - b$, if we first multiply by a, we obtain too great a product by b times the quantity $(a + b)$; if, however, we either subtract b times $(a + b)$, or multiply $(a + b)$ by $-b$, attending to the rule of signs, and add the product, we shall obtain the true result. The latter method is the most usual and correct process in multiplying by negative quantities. Any doubts of the rule are best removed by working examples with numbers, as $(13 + 6) \times (7 - 4)$. Hence the following rule:

Multiply every term of the multiplicand by every term of the multiplier, remembering to place $+$ *before products of quantities with like signs,* *and — before products of quantities with unlike signs. Make all possible reductions, by addition or subtraction of like terms.*

In multiplying it is best, perhaps, first to determine the sign, then the co-efficient, and lastly the letters: and to facilitate the reduction by addition or subtraction, *like quantities* should be placed one under another.

27. The following examples, written as it is usual to arrange similar examples containing a greater number of terms, are of very frequent recurrence. Their products should be written out from memory without any hesitation, when required in any future chapter.

$$\begin{array}{lll}
a+b & a-b & a+b\\
a+b & a-b & a-b\\
\hline
a^2+ab & a^2-ab & a^2+ab\\
\qquad ab+b^2 & \qquad -ab+b^2 & \qquad -ab-b^2\\
\hline
a^2+2ab+b^2 & a^2-2ab+b^2 & a^2 \qquad\quad -b^2
\end{array}$$

28. The importance of these elementary theorems can scarcely be exaggerated. They are constantly reappearing in almost every variety of form throughout the succeeding chapters.

29. In cases where more than two compound quantities are to be multiplied together, first multiply two of them together, and then multiply this product by another of the compound quantities, and this second product again by any of the remaining quantities: and so on. It makes no difference in whatever order the multiplications are made; change them as you please, the final product will be the same.

The multiplication of quantities with postitive, negative, or fractional indices is explained in Art. 82.

EXAMPLES.

(1) Multiply $x^5 + a^5 - a\,x\,(x^3 + a^3)$ by $(x^3 + a^3) + a\,x\,(x + a)$.

$$x^5 + a^5 - a\,x\,(x^3 + a^3) = x^5 - a\,x^4 - a^4\,x + a^5$$
$$x^3 + a^3 + a\,x\,(x + a) = x^3 + a\,x^2 + a^2\,x + a^3$$

$$\begin{array}{l}
x^5 - a\,x^4 - a^4\,x + a^5\\
x^3 + a\,x^2 + a^2\,x + a^3\\
\hline
x^8 - a\,x^7 - a^4x^4 + a^5\,x^3\\
\qquad a\,x^7 - a^2\,x^6 - a^5\,x^3 + a^6\,x^2\\
\qquad\qquad\quad a^2\,x^6 - a^3\,x^5 - a^6\,x^2 + a^7\,x\\
\qquad\qquad\qquad\qquad a^3\,x^5 - a^4\,x^4 - a^7\,x + a^8\\
\hline
x^8 \quad * \quad - a^4\,x^4 \quad * \quad - a^4\,x^4 \quad * \quad + a^8
\end{array}$$

Ans. $x^8 - 2\,a^4\,x^4 + a^8$. (*Matric.* 1838.)

(2) Find the expansion of $(a+b)^2 \cdot (a-b)^5$. (*Matric.* 1839.)

$$\begin{array}{l}
(a-b)^5=a^5-5\,a^4b+10\,a^3b^2-10\,a^2b^3+5\,ab^4-b^5\\
(a+b)^2=a^2+2\,ab+b^2
\end{array}$$

$$\begin{array}{llllllll}
\hline
a^7 & -5\,a^6b & +10\,a^5b^2 & -10\,a^4b^3 & +5\,a^3b^4 & -a^2b^5 & & \\
 & 2\,a^6b & -10\,a^5b^2 & +20\,a^4b^3 & -20\,a^3b^4 & +10\,a^2b^5 & -2\,ab^6 & \\
 & & a^5b^2 & -5\,a^4b^3 & +10\,a^3b^4 & -10\,a^2b^5 & +5\,ab^6 & -b^7\\
\hline
a^7 & -3\,a^6b & +a^5b^2 & +5\,a^4b^3 & -5\,a^3b^4 & -a^2b^5 & +3\,ab^6 & -b^7
\end{array}$$

(3) Expand $(x+a)(x+b)(x+c)$. (*Mat. Ex.* 1842.)

$$\begin{array}{llll}
x+a & & & \\
x+b & & & \\
\hline
x^2+ax & & & \\
 & bx+ab & & \\
\hline
x^2+(a+b)\,x+ab & & & \\
x+c & & & \\
\hline
x^3+(a+b)\,x^2+abx & & & \\
 & cx^2+(a+b)\,cx+abc & & \\
\hline
x^3+(a+b+c)\,x^2+(ab+ac+bc)\,x+abc & & &
\end{array}$$

(4) Find the continued product of $x-a$, $x+a$, $x-a\sqrt{-1}$, $x+a\sqrt{-1}$. (*B. A. Ex.* 1842.)

$$\begin{array}{lllll}
x & +a\sqrt{-1} & & & \\
x & -a\sqrt{-1} & & & \\
\hline
x^2 & +ax\sqrt{-1} & & & \\
 & -ax\sqrt{-1} & -a^2\times-1 & & \\
\hline
x^2 & * & +a^2 & & \\
x & & +a & & \\
\hline
x^3 & +a^2x & +ax^2 & +a^3 & \\
x & -a & & & \\
\hline
x^4 & +a^2x^2 & +ax^3 & +a^3x & \\
 & -a^2x & -ax^3 & -a^3x & -a^4\\
\hline
x^4 & * & * & * & -a^4
\end{array}$$

(5) Multiply $x^{-\frac{3}{4}}+x^{-\frac{1}{2}}y^{-\frac{1}{4}}+x^{-\frac{1}{4}}y^{-\frac{1}{2}}+y^{-\frac{3}{4}}$ (*Mat.* 1847.)

$$\begin{array}{lllll}
x^{-\frac{1}{4}} & -y^{-\frac{1}{4}} & & & \\
\hline
x^{-1} & +x^{-\frac{3}{4}}y^{-\frac{1}{4}} & +x^{-\frac{1}{2}}y^{-\frac{1}{2}} & +x^{-\frac{1}{4}}y^{-\frac{3}{4}} & \\
 & -x^{-\frac{3}{4}}y^{-\frac{1}{4}} & -x^{-\frac{1}{2}}y^{-\frac{1}{2}} & -x^{-\frac{1}{4}}y^{-\frac{3}{4}} & -y^{-1}\\
\hline
x^{-1} & * & * & * & -y^{-1}
\end{array}$$

(6) Multiply $x^{-\frac{2}{3}} + x^{-\frac{1}{3}}y^{-\frac{1}{3}} + y^{-\frac{2}{3}}$ (*Mat.* 1848.)

$$\begin{array}{llll}
 & x^{-\frac{1}{3}} - y^{-\frac{1}{3}} & & \\
\hline
x^{-1} & + x^{-\frac{2}{3}}y^{-\frac{1}{3}} & + x^{-\frac{1}{3}}y^{-\frac{2}{3}} & \\
 & - x^{-\frac{2}{3}}y^{-\frac{1}{3}} & - x^{-\frac{1}{3}}y^{-\frac{2}{3}} & - y^{-1} \\
\hline
x^{-1} & * & * & - y^{-1}
\end{array}$$

DIVISION.

30. As multiplication is a contracted method of addition, so division is a contracted method of subtraction. To divide 39 by 4, or to find how often 4 is contained in 39, proceed by subtracting 4 first from 39; then from the remainder; and again from the second remainder; and so on, until no remainder, or one less than 4 is left. Then the number of subtractions is the number of *times* 4 is contained in 39. In this example 39 is the *dividend*, 4 is the *divisor*, 9 is *called* the *quotient*, (but it is not the true quotient, which is a fraction,) and 3 is the remainder.

31. To illustrate the principles of division as applied to higher numbers, take the example given, *Matric. Exam.*, 1848.

```
347)43375(125
    347
    ———
     867
     694
     ————
     1735
     1735
     ————
```

Here, instead of subtracting 347 separately, we take away at the first subtraction a hundred times 347; at the second subtraction, twenty times 347; and at the third, five times 347. The principle upon which this is done, is, that if any number $p = r + s + t$, then

$$\frac{p}{q} = \frac{r}{q} + \frac{s}{q} + \frac{t}{q}.$$

Thus $\frac{43375}{347} = \frac{34700}{347} + \frac{6940}{347} + \frac{1735}{347} = 100 + 20 + 5 = 125.$

32. Division is the opposite of multiplication, and if the divisor be multiplied by the quotient, and the remainder, if there be one, added to this product, the result ought to be the dividend. With respect to the signs, therefore, $a\,b \div a$ gives undoubtedly the quotient b; and $- a\,b \div a$ gives $- b$, because the quotient multiplied by the divisor must give the dividend, and we have proved

the rule of signs in multiplication to be + into − gives −, (Art. 23,) therefore $+\ a$ (the divisor) into $-\ b$ (the quotient) must give $-\ a\ b$ (the dividend). Again: $-\ a\ b \div (-\ a)$ gives the quotient $+\ b$, because $-\ a$ (the divisor) into $+\ b$ (the quotient) gives $-\ a\ b$ (the dividend). Thus it appears, that the rule of signs in division is the same as in multiplication: viz.

Like signs divided by each other give +
Unlike . . . . −

33. In dividing a compound by a simple quantity, divide each term separately as in the first example below; but if any term of the dividend is not divisible by the divisor, (as in the second example below,) the quotient becomes a fractional quantity.

I. $\dfrac{12\,a - 6\,a\,b}{3\,a} = 4 - 2\,b.$ II. $\dfrac{2\,a^2 - a\,b}{a^2} = 2 - \dfrac{b}{a}.$

34. As an example for dividing a compound by a compound quantity, divide $m\,y + n\,y$ by $m + n$; remembering that the quotient multiplied by the divisor must equal the dividend. Observe, that y must be a part of the quotient to form $m\,y$; and further, that $m + n$ multiplied by y gives $m\,y + n\,y$ the dividend; therefore y is the exact quotient.

35. Again, to divide $x^2 - 5\,x + 6$ by $x - 3$ (*Matric. Exam.* 1849).

First arrange the divisor and dividend as in long division of numbers, and observe that x, the first quantity in the divisor, multiplied by x gives x^2 the first quantity in the dividend. Therefore x must be one of the quantities in the quotient. If now x multiplied by the whole divisor does not equal the whole dividend, subtract, as in long division of numbers, and divide the remainder by the same divisor; and repeat the process until there is no remainder, (as in Ex. 1,) or until it becomes evident that the division will not terminate (as in Ex. 2).

$$\text{I.}\quad \begin{array}{l} x-3)x^2-5\,x+6(x-2. \\ \qquad\ \ x^2-3\,x \\ \hline \qquad\quad -2\,x+6 \\ \qquad\quad -2\,x+6 \\ \hline \end{array} \qquad\qquad \text{II.}\quad \begin{array}{l} 1+x)1 \quad (1-x+x^2-x^3+\&c. \\ \qquad\ 1+x \\ \hline \qquad\ -x \\ \qquad\ -x-x^2 \\ \hline \qquad\qquad x^2 \\ \qquad\qquad x^2+x^3 \\ \hline \qquad\qquad -x^3 \end{array}$$

36. General rule. To facilitate division, arrange both dividend and divisor according to the powers of the letter with the highest

index, place the highest power of both on the extreme left of the two series. Any numerical coefficients or letters common to every term of both dividend and divisor may be struck out before proceeding to the division. Then find how often the first term of the divisor is contained in the first term of the dividend, place this quantity in the quotient and multiply every term of the divisor by it, subtract this product from the dividend, to the remainder bring down as many terms from the dividend as will make its number of terms equal to those in the divisor, repeat this process until all the terms of the dividend are brought down. When there is a remainder and the division can be carried no further, it must be *represented* by a fraction, the remainder as a numerator, and divisor as a denominator.

For the division of quantities with positive, negative, or fractional indices, see Art. 82.

EXAMPLES.

(1) Divide $21x^3 + x^2y - 28x^2z - 53xyz - 10xy^2 + 22y^2z$ by $7x^2 + 5xy - 11yz$.

$$\begin{array}{l}
7x^2+5xy-11yz)21x^3+x^2y-28x^2z-53xyz-10xy^2+22y^2z(3x-2y-4z \\
\qquad\qquad\quad 21x^3+15x^2y-33xyz \\
\hline
\qquad\qquad\qquad -14x^2y-28x^2z-20xyz \\
\qquad\qquad\qquad -14x^2y-10xy^2+22y^2z \\
\hline
\qquad\qquad\qquad\qquad -28x^2z-20xyz+44yz^2 \\
\qquad\qquad\qquad\qquad -28x^2z-20xyz+44yz^2 \\
\hline
\end{array}$$

(2) Divide $(x^3 - 6x^2 + 11x - 6)(x - 4)$ by $x - 1$. (*Mat.* 1842.)

$$\begin{array}{l}
\text{First } x - 1)x^3 - 6x^2 + 11x - 6(x^2 - 5x + 6 \\
\qquad\qquad\quad x^3 - x^2 \\
\hline
\qquad\qquad\qquad -5x^2 + 11x \\
\qquad\qquad\qquad -5x^2 + 5x \\
\hline
\qquad\qquad\qquad\qquad 6x - 6 \\
\qquad\qquad\qquad\qquad 6x - 6 \\
\hline
\end{array}$$

Then $(x^2 - 5x + 6)(x - 4) = x^3 - 9x^2 + 26x - 24$.

Otherwise $(x^3 - 6x^2 + 11x - 6)(x - 4) = (x - 1)(x - 2)(x - 3)(x - 4)$ which divided by $(x - 1)$ gives $(x - 2)(x - 3)(x - 4) = x^3 - 9x^2 + 26x - 24$ as before.

(3) Divide $x^3 - y^2$ by $x^{\frac{1}{2}} - y^{\frac{1}{3}}$. (*Mat.* 1839.)

$$
\begin{array}{l}
x^{\frac{1}{2}} - y^{\frac{1}{3}}\big)\, x^3 - y^2 \,\big(x^{\frac{5}{2}} + x^2 y^{\frac{1}{3}} + x^{\frac{3}{2}} y^{\frac{2}{3}} + x y + x^{\frac{1}{2}} y^{\frac{4}{3}} + y^{\frac{5}{3}} \\
\qquad\qquad\; \underline{x^3 - x^{\frac{5}{2}} y^{\frac{1}{3}}\qquad\quad} \\
\qquad\qquad\qquad x^{\frac{5}{2}} y^{\frac{1}{3}} - y^2 \\
\qquad\qquad\qquad \underline{x^{\frac{5}{2}} y^{\frac{1}{3}} - x^2 y^{\frac{2}{3}}\qquad\quad} \\
\qquad\qquad\qquad\qquad\quad x^2 y^{\frac{2}{3}} - y^2 \\
\qquad\qquad\qquad\qquad\quad \underline{x^2 y^{\frac{2}{3}} - x^{\frac{3}{2}} y\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\quad x^{\frac{3}{2}} y - y^2 \\
\qquad\qquad\qquad\qquad\qquad\quad \underline{x^{\frac{3}{2}} y - x y^{\frac{4}{3}}\qquad\quad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad x y^{\frac{4}{3}} - y^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x y^{\frac{4}{3}} - x^{\frac{1}{2}} y^{\frac{5}{3}}\qquad} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x^{\frac{1}{2}} y^{\frac{5}{3}} - y^2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x^{\frac{1}{2}} y^{\frac{5}{3}} - y^2}
\end{array}
$$

37. Questions on the following criteria of the divisibility of numbers are frequently asked.

1. The divisibility of any number by 2 or by 5 depends upon its unit's figure, because 10 or any number of tens is divisible by 2 and by 5. Hence any figures whatever in the tens', hundreds', &c., places, are divisible by 2 and by 5. Therefore any number is divisible by 2, if its unit's figure be an even number, and by 5 when its unit's figure is 5 or 0.

2. The divisibility of a number by 4 depends upon the unit's and tens' figures, because 100 or any number of hundreds is divisible by 4. Hence, any figures whatever in the hundreds', thousands', &c., places, are divisible by 4: therefore any number is divisible by 4 if its last two figures are divisible by 4.

3. Similarly the divisibility of a number by 8 depends upon the last three figures, because 1000 is divisible by 8. Any number therefore is divisible by 8 if the last three figures are so.

4. The divisibility of a number by 9 or 3 depends upon the sum of its digits; because any number, as 7632, is composed of

$$
\begin{array}{rcrcrcr}
7000 & = & 7 & \times & 999 & + & 7 \\
600 & = & 6 & \times & 99 & + & 6 \\
30 & = & 3 & \times & 9 & + & 3 \\
2 & = & & & & & 2 \\
\hline
7632 & & & & & &
\end{array}
$$

Now 999, 99, and 9, are divisible by 9 and by 3; and the divisibility of the number, therefore, obviously depends upon the remaining quantities, 7, 6, 3, 2, which are the digits composing the number; therefore, any number is divisible by 9 or 3, when the sum of its digits is divisible by 9 or 3.

5. The divisibility of a number by 6 depends upon its being *even* and divisible by 3, because 6 is composed of the factors 2 and 3; and therefore every number divisible by 6, or, which is the same thing, every number that is a multiple of 6, must also be a multiple of the two factors 2 and 3. Any number, therefore, is divisible by 6, when it is even and divisible by 3.

6. Similarly the divisibility of a number by 12 depends on its being divisible by 4 and by 3, because 12 is composed of the factors 4 and 3. Any number, therefore, is divisible by 12 when it is divisible by 4 and by 3.

7. The divisibility of any number by 11 is more difficult to determine than any of the preceding numbers.

Any number, as 6743, is composed of

$$\begin{array}{rcrcrcr}
3 & = & & & & & 3 \\
40 & = & 4 & \times & 11 & - & 4 \\
700 & = & 7 & \times & 99 & + & 7 \\
\underline{6000} & = & 6 & \times & 1001 & - & 6 \\
6743 & & & & & &
\end{array}$$

Now 11, 99, and 1001, are obviously divisible by 11; and the divisibility of the entire number by 11, therefore, depends upon the digits, $3 - 4 + 7 - 6$. If these are divisible by 11, the number is so too. Hence any number is divisible by 11, if, when its digits are taken alternately, and the sum of one series taken from the other, the remainder is 0 or 11.

8. Lastly: a number is divisible by 10 only when its last figure is a cipher. No useful rule has been found to determine the divisibility of a number by 7.

38. The rule for casting out the nines, which is used to verify the product of two numbers, is deduced from the rule to determine the divisibility of a number by 9.

Let N and N' be two numbers: if they are both divisible by 9, then it is evident their product is divisible by 9. But suppose they are not, and let

$$N = 9n + r$$
$$N' = 9n' + r'$$
$$\therefore\ NN' = 81nn' + 9n'r + 9nr' + rr'.$$

Now, both sides of this equation being the same number, and all the quantities on the right side except rr' being evidently divisible by 9, it follows that $N N'$ and $r r'$ will give, when divided by 9, the same remainder. Hence the rule:—Divide successively by 9 the sums of the digits of the multiplicand, multiplier, and product. Then divide the product of the first and second remainders also by 9, and the quotient should be the same as the third remainder.

39. *Prime numbers.*—Every whole number which has no divisor, except unity, is called a prime number.

A number which is exactly divisible by any other number greater than unity, is said to be a composite number.

From the preceding remarks on the conditions of the divisibility of numbers, it is certain that no even number, except 2, can be a prime number. Among odd numbers, the multiples of 3 and 5, are discovered by the tests of Article 37.

If any number is divided by 6, a certain quotient, q, is found, and a remainder of 1, 2, 3, 4, 5, or 0; therefore all numbers may be considered as multiples of 6, with one of the remainders 1, 2, 3, 4, or 5.

Now a multiple of 6, with the remainder 2 or 4, is divisible by 2, and a multiple of 6, with the remainder 3, is divisible by 3. Whence the prime numbers must all be found amongst multiples of 6 + 1, and multiples of 6 + 5. These multiples of 6 may be represented thus:

$$\left.\begin{matrix} 6 \times q + 1 \\ 6 \times q + 5 \end{matrix}\right\} \text{ or } \left.\begin{matrix} 6 \times q + 1 \\ 6 \times q + 6 - 1 \end{matrix}\right\} \text{ or } \begin{matrix} 6 \times q + 1 \\ 6(q+1) - 1 \end{matrix}$$

Let $q + 1 = q'$ and the expression becomes

$$\begin{matrix} 6 \times q + 1 \\ 6 \times q' - 1 \end{matrix}$$

Since the conclusions deduced depend only on the condition that $6 \times q$, and $6 \times q'$, are multiples of 6 without reference to the particular values of q and q', these conclusions must be true when $q = q'$. Whence both of the expressions may be put under the form $6q \pm 1$. This conclusion expressed in words is: all prime numbers greater than 3 have this property, that if they are augmented or diminished by unity, the results are divisible by 6.

But although all prime numbers are comprehended in the formula $6q \pm 1$ (q' being any whole number whatever), yet all the numbers expressed by $6q \pm 1$ are not prime numbers. For the expressions $6 \times q + 1$, $6 \times q + 3$, $6q + 5$, evidently comprehend all odd numbers greater than 5. Of these $6 \times q + 3$, which expresses multiples of 3, is not included in the general formula $6 \times q \pm 1$. This formula, therefore, comprehends all odd numbers which are not multiples of 3. Now the product of any two prime numbers greater than 2 or 3, is an odd number which cannot be divided by 3. This product being an odd number not a multiple of 3, is expressed by the formula $6q \pm 1$, and it is not a prime number.

To find the prime factors of any given number. Let N represent the given number, and a the least prime number by which N is divisible; and let the division of N, and the successive quotients by a be repeated until after n divisions, a quotient N', not divisible by a, is found.

$$\text{Then } N = a^n \times N'.$$

Let b represent the least prime number by which N' can be divided, n' the number of divisions possible by b and N'', the quotient not divisible by b.

$$\text{Then } N' = b^{n'} \times N'',$$

$$\text{And } N = a^n \times N' = a^n \times b^{n'} \times N''.$$

Let c be the prime number which divides N''; let the division of N'' by c be n'' times possible; and the quotient not divisible by c N'''.

$$\text{Then } N'' = c^{n''} \times N''',$$

$$\text{And } N = a^n \times b^{n'} \times N'' = a^n \times b^{n'} \times c^{n''} \,.\, N'''.$$

Continuing this process, a prime number, or some power of a prime number, must at length be obtained for quotient. Dividing as often as possible by this prime number, a last quotient equal to unity is found.

Ex. 1. Resolve 54180 into its prime factors.

$$\begin{array}{r} 2)54180 \\ \hline 2)27090 \\ \hline 3)13545 \\ \hline 3)4515 \\ \hline 5)1505 \\ \hline 7)301 \\ \hline 43 \end{array}$$

$\therefore$ 2, 3, 5, 7, 43, are the prime factors, and $54180 = 2^2 \times 3^2 \times 5 \times 7 \times 43$.

CHAPTER II.

GREATEST COMMON MEASURE, LEAST COMMON MULTIPLE, AND VULGAR AND DECIMAL FRACTIONS.

GREATEST COMMON MEASURE.

40. A NUMBER is a common measure of any set of numbers, when it will divide each of them without a remainder; thus 2, 3, 4, 6, and 12, will divide both 36 and 48, and 12 is the largest number that will divide them both, or it is their *greatest common measure*, for convenience usually written *g. c. m.* Where a common measure of a set of quantities is wanted, any common measure will do, but the greatest is the most convenient, and hence the occasion for a rule to find the greatest common measure.

41. To investigate a rule for finding the *g. c. m.* applicable alike to numbers and algebraic quantities—

Let a and b be two quantities, of which a is the greater, and let b be contained p times in a with a remainder c, let c be contained q times in b with a remainder d, and let d be contained r times in c with no remainder.

$$\begin{array}{l}
b)\;a\;(p \\
\;\;\;p\,b \\
\hline
\;\;\;c)\;b\;(q \\
\;\;\;\;\;\;q\,c \\
\hline
\;\;\;\;\;\;d)\;c\;(r \\
\;\;\;\;\;\;\;\;\;r\,d \\
\hline
\end{array}$$

Then d is the *g. c. m.* of a and b.

For since $c - r\,d = 0 \therefore c = r\,d$

And $b - q\,c = d \therefore b = q\,c + d = q\,r\,d + d = (q\,r + 1)\,d$. Again $a - p\,b = c$. $\therefore a = p\,b + c = (p\,q\,r + p + r)\,d \therefore d$ measures b and a, by the units contained in $q\,r + 1$ and $p\,q\,r + p + r$ respectively, and is therefore a common measure of a and b.

It is also their *greatest* common measure, for if not, let D be their greatest common measure, and let

$$a = m\,D \text{ and } b = n\,D$$

$$\text{Then } c = a - p\,b = m\,D - p\,n\,D = (m - p\,n)\,D$$

$$\begin{aligned}\text{And } d = b - q\,c &= n\,D - (q\,m - p\,q\,n)\,D \\ &= (n - q\,m + p\,q\,n)\,D\end{aligned}$$

$\therefore D$ measures d by the units contained in the quantities $n - q$

$m + p\,q\,n$, that is D the greater measures d, the less, which is absurd. $\therefore$ D is not the *g. c. m.* of a and b, and in the same way it may be proved that no other quantity than d is the *g. c. m.* of a and b. $\therefore$ d is their *g. c. m.*

42. The preceding reasoning proves that, by the following rules, you must obtain the *g. c. m.* of any two numbers.

Divide the greater number by the less, and the preceding divisor by the last remainder, and continue the operation till there is no remainder, then will the last divisor be the greatest common measure.

Ex. 1.	247)570(2	Ex. 2.	78624	102232	1
	494		70824	78624	
	76)247(3		7800	23608	3
	228		624	23400	3
	19)76(4		1560	208	37
	76		1456	208	2
			104		

Ex. 1, *g. c. m.* 19; 2, *g. c. m.* 104.

43. To find the *g. c. m.* of three numbers, find the *g. c. m.* of any two, and of this, and the third.

Let a, b, c be any numbers or algebraic quantities, and let d be the *g. c. m.* of a and b, then the *g. c. m.* of d and c shall be the *g. c. m.* of a, b, and c. Since d is the *g. c. m.* of a and b, *every* other common measure of a and b is contained in d, and $\therefore$ *every c. m.* of a, b, and c must be contained in d, because the *g. c. m.* of three numbers or quantities is always either the same or less than the *g. c. m.* of two of these numbers or quantities. Hence it follows that the *g. c. m.* of d and c is also the *g. c. m.* of a, b, and c.

Similarly to find the *g. c. m.* of four numbers or quantities, find the *g. c. m.* of the first, second, and third, and of that and the fourth, and in the same manner the *g. c. m.* of any series of numbers or quantities may be determined by a continuation of the process above.

44. In reducing fractions to their lowest terms instead of dividing numerator and denominator by their *g. c. m.*, it is often more expeditious to take any measure that may be found by inspection, according to the principles laid down, Art. 37, and divide numerator and denominator by it, and again divide numerator and denominator of this second fraction by a new measure found as before by inspection, and so on, till the fraction is reduced to its lowest terms.

45. If a quantity measure each of two others, it will also measure their sum or difference.

Let P measure Q by the units in m, and let it measure R by the units in n:

$$\text{Then } Q = m\,P \text{ and } R = n\,P.$$
$$\therefore Q \pm R = (m \pm n)\,P,$$

or P measures $Q \pm R$ by the units in $m \pm n$.

It will be frequently useful to remember this property of two numbers or quantities. Thus $\frac{53}{58}$ is in its lowest terms because 5, their difference, will not divide either number, nor will any number divide 5.

46. The rule for finding the *g. c. m.* of two or more numbers needs no additional modification, but in the case of algebraical quantities, it will be necessary in complicated expressions, to simplify them before applying the rule, in order to avoid fractional coefficients, and also to modify the remainders and divisors during the process. These ends are gained in two ways.

1° By multiplying or dividing either quantity, or any divisor or remainder, by any number or quantity.

2° Any factor common to both divisor and dividend may be struck out; but as it is a factor of the *g. c. m.*, it must be multiplied afterwards into the *g. c. m.* of the remaining quantities.

Ex. Find the *g. c. m.* of

$3ax^2-38ax+119a$ and $6b^2x^3-114b^2x^2+714b^2x-1470b^2$.

Omitting a the common factor of the first, and also $2b^2$ the common factor of the second quantity, they become when arranged as divisor and dividend,

$$\begin{array}{r|l}
3x^2-38x+119 & 3x^3-57x^2+357x-735\,(x+19 \\
 & \underline{3x^3-38x^2+119x} \\
 & -19x^2+238x-735 \\
 & \underline{-3} \\
 & 57x^2-714x+2205 \\
 & \underline{57x^2-722x+2261} \\
 & 8x-56
\end{array}$$

Rejecting the common factor 8 from the remainder,

$$\begin{array}{r|l}
x-7 & 3x^2-38x+119\,(3x-17 \\
 & \underline{3x^2-21x} \\
 & -17x+119 \\
 & \underline{-17x+119}
\end{array}$$

Hence $x-7$ is the *g. c. m.* required.

EXAMPLES.

(1) Find the *g. c. m.* of $x^5 - y^5$ and $x^2 - y^2$. (*Mat.* 1838.)

$$\begin{array}{r|l}
x^2 - y^2 & x^5 - y^5 \quad (x^3 + x y^2 \\
 & x^5 - x^3 y^2 \\ \hline
 & x^3 y^2 - y^5 \\
 & x^3 y^2 - x y^4 \\ \hline
y^4 & x y^4 - y^5 \\ \hline
 & x - y, \text{ the } g.\ c.\ m. \text{ required.}
\end{array}$$

(2) Find the *g. c. m.* of $\dfrac{3 x^2 + 12 x + 9}{x^5 + 5 x^3 + 6}$. (*Mat.* 1839.)

Dividing the numerator by 3, we have $x^2 + 4 x + 3$, then—

$$\begin{array}{r|l}
x^2 + 4 x + 3 & x^5 + 5 x^3 + 6 \quad (x^3 - 4 x^2 + 18 x - 60 \\
 & x^5 + 4 x^4 + 3 x^3 \\ \hline
 & -4 x^4 + 2 x^3 + 6 \\
 & -4 x^4 - 16 x^3 - 12 x^2 \\ \hline
 & 18 x^3 + 12 x^2 + 6 \\
 & 18 x^3 + 72 x^2 + 54 x \\ \hline
 & -60 x^2 - 54 x + 6 \\
 & -60 x^2 - 240 x - 180 \\ \hline
186 & 186 x + 186 \\ \hline
 & x + 1
\end{array}$$

$$\begin{array}{r|l}
x + 1 & x^2 + 4 x + 3 \quad (x + 3 \\
 & x^2 + x \\ \hline
 & 3 x + 3 \\
 & 3 x + 3 \\ \hline
\end{array}$$

$\therefore\ x + 1$ is the *g. c. m.*

(3) Find the *g. c. m.* of $\dfrac{x^3 - 6 x^2 - 37 x + 210}{x^3 + 4 x^2 - 47 x - 210}$. (*B.A.* 1842.)

$$\begin{array}{r|l}
x^3 + 4 x^2 - 47 x - 210 & x^3 - 6 x^2 - 37 x + 210 \quad (1 \\
 & x^3 + 4 x^2 - 47 x - 210 \\ \hline
-10 & -10 x^2 + 10 x + 420 \\ \hline
 & x^2 - x - 42
\end{array}$$

$$\begin{array}{r|l}
x^2 - x - 42 & x^3 + 4 x^2 - 47 x - 210 \quad (x + 5 \\
 & x^3 - x^2 - 42 x \\ \hline
 & 5 x^2 - 5 x - 210 \\
 & 5 x^2 - 5 x - 210 \\ \hline
\end{array}$$

$\therefore\ x^2 - x - 42$ is the *g. c. m.*

LEAST COMMON MULTIPLE.

47. Suppose two numbers, 24 and 36, given to find their least common multiple, (generally written *l. c. m.*)

First divide them both by their *g. c. m.* 12, and since their *l. c. m.* must be such a number as will contain them both exactly, it must be a multiple of their *g. c. m.* 12. It must also contain the factors 12 and 2 to make 24, and 12 and 3 to make 36; but 2 and 3 are the least possible multiples of 12 which compose the numbers 24 and 36, and the *l. c. m.* required must contain them both, and also 12, and it is therefore the product of the three.

48. Rule. To find the least common multiple of two numbers, find their greatest common measure, and divide either of them by it and multiply the other by this quotient.

To find the least common multiple of three or more numbers.

Rule. Find the least common multiple of the first two numbers; then the least common multiple of that multiple and the third number, and so on. The last common multiple so found will be the least common multiple required.

When the least common multiple of several numbers is required, the most convenient practical method is that given by the following

Rule. Arrange the numbers in a line from left to right, with a comma between each. Divide those numbers which have a common measure by that common measure, and place the quotients so obtained, and the undivided numbers, in a line beneath, separated as before. Proceed in the same way with the second line, and so on with those that follow, until a row of numbers is obtained in which there are no two numbers which have any common measure greater than unity. Then the continued product of all the divisors and the numbers in the last line will be the least common multiple required.

NOTE.—It will, in general, be found advantageous to begin with the lowest prime number 2 as a divisor, and to repeat this as often as can be done, and then to proceed with the prime numbers 3, 5, &c., in the same way.

Ex. 1. Find the least common multiple of 8, 12, 30, 42, 60.

2) 8, 12, 30, 42, 60.
2) 4, 6, 15, 21, 30.
3) 2, 3, 15, 21, 15.
5) 2, 1, 5, 7, 5.
2, 1, 1, 7, 1.

$$\therefore \text{the } l.\,c.\,m. = 2 \times 2 \times 3 \times 5 \times 2 \times 7 = 840.$$

Ex. 2. Find the *l. c. m.* of $12\,x^2y^3z^4$ and $8\,y^5z^3x^2$.

The *g. c. m.* of $12\,x^2y^3z^4$ and $8\,y^5z^3x^2$ is evidently $4\,x^2y^3z^3$,

$$\therefore l.\,c.\,m. = \frac{12\,x^2y^3z^4}{4\,x^2y^3z^3} \times 8\,y^5z^3x^2 = 3\,z \times 8\,y^5z^3x^2 = 24\,x^2y^5z^4.$$

Ex. 3. Find the *l. c. m.* of $x^2 - a^2$ and $(x - a)^2$.

$$x^2 - a^2 = (x + a)(x - a) \text{ and } (x - a)^2 = (x - a)(x - a)$$

$$\therefore x - a \text{ is the } g.\,c.\,m. \text{ of } x^2 - a^2 \text{ and } (x - a)^2$$

$$\therefore l.\,c.\,m. = \frac{(x + a)(x - a)}{x - a} \times (x - a)(x - a) = x^3 - ax^2 - a^2x + a^3.$$

Ex. 4. Find the *l. c. m.* of $(a^2x - ax^2)^2$ and $ax\,(a^2 - x^2)^2$.

$$(a^2x - ax^2)^2 = \{\,ax\,(a - x)\,\}^2 = a^2x^2\,(a - x)^2$$

$$\text{and } ax\,(a^2 - x^2)^2 = a\,x\,(a + x)^2\,(a - x)^2$$

$$\therefore g.\,c.\,m. = ax\,(a - x)^2$$

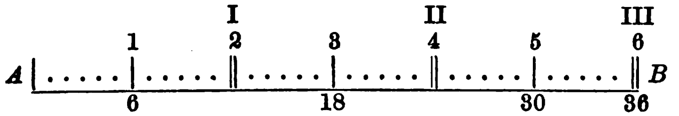
$$\text{and } l.\,c.\,m. = \frac{a^2x^2\,(a - x)^2}{a\,x\,(a - x)^2} \times a\,x\,(a^2 - x^2)^2 = a^2x^2\,(a^2 - x^2)^2.$$

VULGAR FRACTIONS.

49. Suppose unity equally divided into parts, a certain number of these parts form a fraction; thus $\frac{3}{4}$ represents unity divided into four equal parts, and three of them form the fraction.

50. In integer numbers, when *concrete* quantities are spoken of, the digits employed indicate the number of the particular *denomination;* also in fractions, the number expressing the parts into which unity is divided, is called the *denominator,* and the digits expressing the *number* of these divisions in any given fraction, is the *numerator.*

51. The principal properties of fractions may be well illustrated by taking a line as a unit, and supposing it divided into three, six, and thirty-six equal parts.



1st. Where the numerator of a fraction is equal to its denominator, the fraction itself represents unity; thus—

$$\frac{3}{3},\ \frac{6}{6},\ \frac{36}{36}$$

are all equal to unity; because the line *A B* is divided into 3, 6, and 36 equal divisions, and if 3, 6, and 36 of these divisions be successively taken, the whole unit will be taken in each instance; therefore generally $\frac{x}{x}$ represents unity.

2nd. It follows from the preceding, that all fractions having numerators less than their denominators, have a value less than unity. Such fractions are called *real* or *proper* fractions. For if we suppose the line *A B* to be divided as before, the fractions

$$\frac{2}{3},\ \frac{5}{6},\ \frac{24}{36}$$

will each represent less than the whole line.

52. When the numerator is greater than the denominator, it is called an *improper* fraction, and its value is greater than unity. To prove this: Divide a line into two parts, and take three of these parts. You thus take the whole line and one-half of the line, or $\frac{3}{2} = 1\frac{1}{2}$. In like manner $\frac{4}{3}$ of a line, is a line divided into three parts and four of them taken, or $\frac{4}{3} = 1\frac{1}{3}$. In the same way $\frac{5}{3} = 1\frac{2}{3}$, $\frac{7}{3} = 2\frac{1}{3}$, and $\frac{9}{3} = 3$. $1\frac{2}{3}$, $2\frac{1}{3}$, &c., are called mixed numbers.

53. To resolve an improper fraction to an integer or mixed number: Divide the numerator by the denominator and to the quotient annex the remainder as a fraction, with the divisor for a denominator. Thus—

$$\frac{31}{4} = \frac{7 \times 4 + 3}{4} = 7\frac{3}{4} \qquad \frac{53}{15} = \frac{15 \times 3 + 8}{15} = 3\frac{8}{15}$$

54. To reduce a mixed number into an improper fraction: Multiply the integer number by the denominator of the fraction, and to the product add the numerator for a new numerator, retaining the same denominator. Thus—

$$7\frac{3}{4} = \frac{7 \times 4 + 3}{4} = \frac{31}{4} \qquad 3\frac{8}{15} = \frac{3 \times 15 + 8}{15} = \frac{53}{15}$$

This process is evidently the reverse of the preceding.

55. The value of a fraction is not altered by multiplying its numerator and denominator by the same number. This is evident if we only consider that $\frac{3}{4}$ of a pound sterling, or three crowns are the same part of one pound as $\frac{3 \times 5}{4 \times 5}$ or $\frac{15}{20}$ of a pound, or 15 shillings.

Again refer to the line *A B*, and observe that $\frac{2}{3}$ of the line is the same portion of it as $\frac{2 \times 12}{3 \times 12}$ or $\frac{24}{36}$ of it. Therefore generally,

$$\frac{x}{y}, \frac{2x}{2y}, \frac{3x}{3y}, \frac{4x}{4y}, \&c., \frac{nx}{ny},$$

are each equal to one another.

56. From this conclusion it follows immediately, that the numerator and denominator of a fraction may be both divided by the same number, and the fraction will remain unaltered, because it has been just shown that $\frac{x}{y} = \frac{nx}{ny}$, here n is any number we please, and dividing the latter fraction by n we obtain the former.

57. We can thus represent the same fraction in an almost infinite number of ways; for each of the following fractions is equal to 2:—

$$\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \&c. \frac{2n}{n}$$

And the following, all have a common value, 3:—

$$\frac{3}{1}, \frac{6}{2}, \frac{9}{3}, \frac{12}{4}, \&c. \frac{3n}{n}$$

So also the fractions

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \&c. \frac{n}{2n}$$

are also equal.

58. Of all the nearly infinite variety of forms in which a fraction may be thus represented, any one of which may be used for any other without error, it cannot be doubted but that form which is composed of the smallest numbers is most convenient for use, and most readily understood. When a fraction is thus reduced to its simplest form, it is said to be in its *lowest terms*.

To reduce a fraction to its lowest terms it is necessary to find the *g. c. m.* of its numerator and denominator, and dividing both by this number, the quotients can then have no common measure except unity, the fraction is therefore in its lowest terms.

59. It is sometimes convenient to consider the numerator of a fraction as the *dividend*, and the denominator, the *divisor*; thus $\frac{2}{3}$ as equal to $2 \div 3$, or as representing $\frac{1}{3}$ of 2 *units* rather than $\frac{2}{3}$ of 1 *unit*. The result is the same in both cases. For instance, $\frac{2}{3}$ of 1 yard and $\frac{1}{3}$ of 2 yards are both 2 feet.

60. For this reason any whole number may be represented as a fraction by placing unity for its denominator.

61. It is evident that of two fractions which have the same denominator, the greater has the greater numerator; for a larger number of the *same* portions of a unit must be more of the unit than a smaller number of them.

62. Of two fractions which have the same numerator the greater has the less denominator, because any number of the equal parts of

a unit must be greater than the *same* number of more minute parts of a unit.

63. Therefore, by increasing the denominator of a fraction we lessen its value; thus $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. $\frac{1}{10^n}$ are fractions which diminish at every step, and approach to nothing; but as no finite number is so great as to leave the parts into which it divides unity without magnitude, therefore no finite number in the denominator of a fraction will reduce it to nothing. However, as every increase of the denominator diminishes the fraction and brings it nearer to nothing, a fraction with unity in its numerator, and the symbol for infinity in its denominator, is said to be equal to nothing; thus—

$$\frac{1}{\alpha} = 0.$$

64. Similarly, by decreasing the denominator we augment the value of a fraction and approach infinity; but yet no number can be sufficiently small to enable us to attain infinity; yet because as the number in the denominator decreases the fraction increases, therefore unity or any other given number divided by nothing is used to represent infinity.

$$\frac{1}{0} = \alpha$$

ADDITION AND SUBTRACTION OF FRACTIONS.

65. When two or more fractions have a common denominator we can more readily compare them and find their sum or difference. To add them we have only to add their numerators and suffix the common denominator. Suppose, for instance, we wish to add $\frac{4}{11}$ to $\frac{6}{11}$. Here, in both cases, unity has been divided into 11 equal parts, and the first fraction consists of 4, and the second of 6; of those parts, therefore, the sum must consist of $4+6$ or 10 of them, *i. e.*,

$$\frac{4}{11} + \frac{6}{11} = \frac{10}{11}.$$

66 The difference of two fractions which have a common denominator is obtained by subtraction between their numerators and suffixing the common denominator. For example, $\frac{6}{11} - \frac{4}{11}$. Here, again, unity has been divided, in both cases, into 11 equal parts, and the difference between 6 and 4 of those parts must be $6 - 4$ or 2 of the 11 equal parts into which unity has been divided, *i. e.*

$$\frac{6}{11} - \frac{4}{11} = \frac{2}{11}.$$

67. When two or more fractions have not a common denominator, we can always reduce them to a common denominator without altering their value, by multiplying all the denominators together for a new denominator, and the numerator of each fraction into all the denominators except its own; thus—

$$\frac{1}{2}, \frac{2}{3}, \frac{7}{9}, \text{ are respectively equal to } \frac{27}{54}, \frac{36}{54}, \frac{42}{54},$$

and these fractions can now be added and subtracted the same as whole numbers, by adding and subtracting their numerators only; they are not however reduced to their *lowest* common denominator, and therefore they are not in the *most* convenient form to be used in calculation.

68. To reduce two or more fractions to their lowest common denominator; First find the least common multiple of their denominators for a common denominator. Divide the common denominator by the denominator of each fraction, and multiply the quotients thus found into each numerator and denominator respectively. To reduce in this manner $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{9}$ to their least common denominator. First, we find the least common multiple to be 18, and the quotients to be respectively used as multipliers, 9, 6, and 2, and the fractions themselves thus become

$$\frac{1\times 9}{2\times 9} = \frac{9}{18}, \frac{2\times 6}{3\times 6} = \frac{12}{18}, \frac{7\times 2}{9\times 2} = \frac{14}{18}.$$

Here the fractions are not altered in value, by having their numerators and denominators multiplied by the same number; and as 18 is the least common multiple of their denominators it is the (least or) lowest common denominator.

69. To add or subtract a fraction to or from an integer, we have no more to do than to change the integer into a fraction with the same denominator as the given fraction, by first supposing it a fraction with unity for its denominator, Art. 60, and then reducing the two fractions to a common denominator. If, for example, it were required to add $\frac{5}{6}$ to 3. First reduce 3 to a fraction with a denominator 6; thus—

$$\frac{3\times 6}{1\times 6} = \frac{18}{6}, \text{ then } \frac{5}{6} + \frac{18}{6} = \frac{23}{6}.$$

In the same way is $\frac{1}{4}$ subtracted from 1. First reduce 1 to a fraction with the denominator 4; thus—

$$\frac{4}{4} - \frac{1}{4} = \frac{3}{4}.$$

70. To find the sum or difference of two or more fractions, therefore, the rule is

Reduce the given fractions to their least common denominator, if necessary, and take the sum or difference (as required) of their numerators.

EXAMPLES.

(1) Of the fractions $\frac{2}{3}, \frac{5}{7}$ which is the greater? (*Mat.* 1838.)

$$\frac{5}{7} \text{ is } > \frac{2}{3} \because \frac{5}{7} = \frac{15}{21} \text{ and } \frac{2}{3} = \frac{14}{21}.$$

(2) Add together the fractions $\frac{11}{17}, \frac{31}{51}, \frac{266}{357}, \frac{5}{13}, \frac{24}{39}$. (*M.* 1838.)

$$\begin{array}{r|l} 3) & 17,\ 51,\ 357,\ 13,\ 39, \\ \hline 13) & 17,\ 17,\ 119,\ 13,\ 13, \\ \hline 17) & 17,\ 17,\ 119,\ 1,\ 1, \\ \hline & 1,\ 1,\ 7,\ 1,\ 1, \end{array}$$

$$\therefore l.\,c.\,m = 3 \times 13 \times 17 \times 7 = 4641.$$

And the fractions become respectively,

$$\frac{11 \times 273}{17 \times 273} = \frac{3003}{4641} \left(\text{since } \frac{4641}{17} = 273\right)$$

$$\frac{31 \times 91}{51 \times 91} = \frac{2821}{4641} \left(\text{since } \frac{4641}{51} = 91\right)$$

$$\frac{266 \times 13}{357 \times 13} = \frac{3458}{4641} \left(\text{since } \frac{4641}{357} = 13\right)$$

$$\frac{5 \times 357}{13 \times 357} = \frac{1785}{4641} \left(\text{since } \frac{4641}{13} = 357\right)$$

$$\frac{24 \times 119}{39 \times 119} = \frac{2856}{4641} \left(\text{since } \frac{4641}{39} = 119\right)$$

$$\therefore \text{Sum} = \frac{3003 + 2821 + 3458 + 1785 + 2856}{4641}$$

$$= \frac{13923}{4641} = 3. \quad \textit{Ans.}$$

(3) Find the difference between $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$ and $\frac{1}{3} + \frac{1}{5} + \frac{1}{7}$.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} = \frac{6}{12} + \frac{3}{12} + \frac{2}{12} = \frac{11}{12}$$

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} = \frac{35}{105} + \frac{21}{105} + \frac{15}{105} = \frac{71}{105}$$

$$\text{and } \frac{11}{12} - \frac{71}{105} = \frac{385}{420} - \frac{284}{420} = \frac{101}{420}. \quad \textit{Ans.} \quad (\textit{M.}\ 1839.)$$

(4) Reduce $\frac{2}{3} + \frac{4}{5} + \frac{6}{7} - \frac{8}{9}$. (*M.* 1842.)

$$\text{First } \frac{2}{3} + \frac{4}{5} + \frac{6}{7} = \frac{70}{105} + \frac{84}{105} + \frac{90}{105} = \frac{244}{105}$$

$$\text{then } \frac{244}{105} - \frac{8}{9} = \frac{2196 - 840}{945} = \frac{1356}{945} = \frac{452}{315} \quad \textit{Ans.}$$

(5) Reduce $\dfrac{1}{2 + \dfrac{3}{4 + \frac{5}{6}}}$ (*M.* 1842.)

$$\frac{1}{2 + \dfrac{3.}{4 + \frac{5}{6}}} = \frac{1}{2 + \dfrac{3}{\frac{29}{6}}} = \frac{1}{2 + \frac{18}{29}} = \frac{1}{\frac{76}{29}} = \frac{29}{76}. \quad \textit{Ans.}$$

(6) Add $\frac{a - b}{2}$ to $\frac{a + b}{2}$. (*M.* 1843)

$$\frac{a - b}{2} + \frac{a + b}{2} = \frac{a}{2} - \frac{b}{2} + \frac{a}{2} + \frac{b}{2} = a.$$

(7) From $\frac{3a - 2b}{4}$ take $\frac{2a - 3b}{5}$. (*M.* 1845.)

$$\frac{3a - 2b}{4} - \frac{2a - 3b}{5} = \frac{5(3a - 2b) - 4(2a - 3b)}{20}$$

$$= \frac{15a - 10b - 8a + 12b}{20} = \frac{7a + 2b}{20}.$$

(8) From $\frac{a + b - 2c}{3}$ take $\frac{2a - b + c}{4}$. (*M.* 1846.)

$$\frac{a + b - 2c}{3} - \frac{2a - b + c}{4} = \frac{4(a + b - 2c) - 3(2a - b + c)}{12}$$

$$= \frac{4a + 4b - 8c - 6a + 3b - 3c}{12} = \frac{-2a + 7b - 11c}{12}.$$

(9) Reduce $\frac{a + b}{a - b} + \frac{a - b}{a + b}$. (*M.* 1846, and *B.A.* 1848.)

$$\frac{a + b}{a - b} + \frac{a - b}{a + b} = \frac{(a + b)^2 + (a - b)^2}{a^2 - b^2}.$$

$$= \frac{a^2 + 2\cancel{a}b + b^2 + a^2 - 2\cancel{a}b + b^2}{a^2 - b^2} = \frac{2(a^2 + b^2)}{a^2 - b^2}.$$

(10) From $\frac{a}{a - x}$ take $\frac{a}{a + x}$ (*M.* 1850.)

$$\frac{a}{a - x} - \frac{a}{a + x} = \frac{a(a + x) - a(a - x)}{a^2 - x^2}$$

$$= \frac{a^2 + ax - (a^2 + ax)}{a^2 - x^2} = \frac{2ax}{a^2 - x^2}.$$

(11) Reduce $\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^3 - x^2 y}{y^3 - x^2 y}$ to its most simple equivalent form, and discuss the case when $x = 1, y = 1$. (*B.A.* 1850.)

$$\frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^3 - x^2 y}{y^3 - x^2 y}$$

$$\text{Or } \frac{x+y}{y} - \frac{2x}{x+y} - \frac{x^3 - x^2 y}{x^2 y - y^3}$$

$$= \frac{(x+y)^2 - 2xy}{y(x+y)} - \frac{x^2(x-y)}{y(x^2 - y^2)}$$

$$= \frac{x^2 + y^2}{y(x+y)} - \frac{x^2(x-y)}{y(x-y)(x+y)}$$

$$= \frac{x^2 + y^2}{y(x+y)} - \frac{x^2}{y(x+y)}$$

$$= \frac{y^2}{y(x+y)} = \frac{y}{x+y}.$$

When $x = 1$ and $y = 1$, this becomes $\frac{1}{2}$, a result which could not have been obtained at once from the given expression, because in the term $-\frac{x^3 - x^2 y}{x^2 y - y^3}$ the numerator and denominator have each a factor $x - y = 0$.

Expunging this, the expression becomes

$$\frac{x+y}{y} - \frac{2x}{x+y} - \frac{x^2}{xy + y^2} = 2 - 1 - \frac{1}{2} = \frac{1}{2}, \text{ as before.}$$

MULTIPLICATION AND DIVISION OF FRACTIONS.

71. The multiplication of a fraction by a *whole number* is effected by multiplying the number into the numerator and leaving the denominator unchanged; for, ¼ multiplied by 4 means

$$\tfrac14 \text{ taken 4 times which is } \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{4}{4} \text{ or } 1,$$

$$\text{and similarly } \frac{4}{9} \times 3 = \frac{12}{9} = 1\frac{1}{3}.$$

72. In the second instance, however, the fraction we obtain on multiplying by 3 is not in its lowest terms, for dividing both terms by 3 gives $\frac{4}{3}$, but we can at one step obtain the same result if we at first divide the denominator of the fraction $\frac{4}{9}$ by 3. Therefore when an integer number, which is to be used as a multiplier of a fraction, exactly divides the denominator, we obtain a simpler fraction, and diminish the labour by dividing the denominator, instead of multi-

plying the numerator. Therefore, generally, in multiplying a fraction $\frac{x}{y}$ successively by the whole numbers x and y, the products are,

$$\frac{x}{y} \times x = \frac{x^2}{y}, \text{ and } \frac{x}{y} \times y = x.$$

In the first case we *multiply the numerator*, and in the second, *divide the denominator.*

Hence, to multiply a fraction by an integer, the rule is

Multiply the numerator, or, where practicable, divide the denominator.

73. Again, if we have to *divide* the fraction $\frac{3}{5}$ by an integer number 3, the question asked is, How often is 3 contained in $\frac{3}{5}$? and the answer obviously is, *no times*, because the less cannot contain the greater, but $\frac{1}{5}$ of *a time*, because $\frac{1}{5}$ taken 3 times is $\frac{1}{5}+\frac{1}{5}+\frac{1}{5}=\frac{3}{5}$.

Hence the rule is, Divide the numerator by the whole number and leave the denominator unchanged.

74. This rule can be readily applied when the integer number divides exactly the numerator of the fraction, but not where it leaves a remainder. In the latter case, however, we can multiply both terms of the fraction, without altering its value, by such a number as will make its numerator exactly divisible; thus the numerator of $\frac{2}{5}$ is not, but that of $\frac{6}{15}$ is, divisible by 3, and the quotient is $\frac{2}{15}$; but this result might have been obtained at first by multiplying the denominator by 3.

Therefore, generally, in dividing a fraction $\frac{x}{y}$ successively by x and y, the quotients are,

$$\frac{x}{y} \div x = \frac{1}{y}, \text{ and } \frac{x}{y} \div y = \frac{x}{y^2}.$$

In the first case we *divide the numerator*, and in the second *multiply the denominator.*

Hence, to divide a fraction by an integer, the rule is

Divide the numerator, where practicable, or multiply the denominator.

75. In multiplying by a *fraction*, it is particularly important to remember that we use the word multiplication in a different sense from the one employed in speaking of whole numbers. In multiplying any quantity by a whole number, we take that quantity a certain *number of times*, as the etymology of the word indicates; but in multiplying a quantity by a fraction, we do not take it any *number*

of times, but only certain *parts of a time;* this process is, however, called multiplication, though it is in reality division, and multiplying 6 by $\frac{1}{2}$ means taking 6 one-half of a time, or virtually dividing it by 2.

In like manner, multiplying any quantity by $\frac{2}{3}, \frac{3}{4}, \frac{5}{6}$, or any other real fraction, we take only a certain part or number of parts and not the whole quantity. Premising that this important distinction is clearly understood, we proceed to the development of the rule for multiplication by fractions.

76. In order to multiply $\frac{3}{4}$ by $\frac{2}{3}$ we must remember (Art. 59) that $\frac{2}{3}$ means 2 divided by 3. Now, if we first multiply $\frac{3}{4}$ by 2 we have $\frac{3 \times 2}{4}$, which is evidently too much, as we should only have multiplied by 2 after it had been divided by 3; if, however, we now divide the product by 3 we shall have the true result; hence, $\frac{3 \times 2}{4}$ divided by 3, which we have shown in Art. 74 may be done by multiplying the denominator by 3, gives $\frac{3 \times 2}{4 \times 3}$, which truly represents the multiplication of $\frac{3}{4}$ by $\frac{2}{3}$, and thus generally

$$\frac{m}{x} \times \frac{n}{y} = \frac{m\,n}{x\,y}.$$

77. The rule, therefore, for the multiplication of fractions is: Multiply all the numerators together for a new numerator, and all the denominators for a new denominator.

78. Before investigating the rule for *division* by a fraction, we will first point out an extension of meaning in the use of the word similar to that explained in multiplication. In dividing by a whole number, we ascertain how *many times* one number, *the divisor*, is contained in a larger number, *the dividend;* and the number obtained, the quotient, is always *less* than the dividend. In dividing by a fraction, we ascertain how many times or *parts of a time*, one *part* of a number, or fraction, is contained in *any* given quantity (greater or less than the divisor), and we obtain a quotient always *greater* than the dividend. This process, therefore, which is called division, is in reality multiplication; for example, to divide 6 by $\frac{1}{2}$ means, to ascertain how many halves are contained in 6, which is in fact tantamount to multiplying by 2.

79. It will be seen from the above, and from Art. 75, that multiplication by fractions resembles division by whole numbers, and division by fractions resembles multiplication by whole numbers. Still, it is not correct to say that in fractions every multiplication is

a division, and every division a multiplication, because multiplying 6 by 2 and dividing 6 by $\frac{1}{2}$ are different processes, though they produce the same result, and the same may be said of dividing 6 by 2 and multiplying by $\frac{1}{2}$.

80. In dividing one fraction by another, if both fractions have the same denominator, the division is confined to the numerators; thus, to divide $\frac{4}{5}$ by $\frac{2}{5}$, we must ascertain how many times 2 is contained in 4, and must solve the question just in the same way as if it were required to find how many times £2 is contained in £4; the answer to both is $\frac{4}{2}$, or 2.

In like manner, $\frac{2}{5}$ divided by $\frac{3}{5}$ is $\frac{2}{3}$, and $\frac{7}{10}$ divided by $\frac{8}{10}$ as well as $\frac{7}{100}$ divided by $\frac{8}{100}$ are both $\frac{7}{8}$, therefore generally

$$\frac{x}{z} \div \frac{y}{z} = \frac{x}{y}.$$

81. But if both the fractions have not the same denominator, we may reduce them to a common denominator, and then apply the same rule, thus: $\frac{x}{m} \div \frac{y}{n}$ are of the same value as $\frac{n\,x}{m\,n} \div \frac{m\,y}{m\,n}$, and the quotient derived from the latter is, $\frac{n\,x}{m\,y}$, but this result, it should be observed, might have been obtained more easily by transposing the terms of the divisor, and using the fraction thus changed as a multiplier, thus:—

$$\frac{x}{m} \div \frac{y}{n} = \frac{x}{m} \times \frac{n}{y} \text{ or, } \frac{x\,n}{m\,y}.$$

It is from this last process that we obtain the rule for division by fractions: *Invert the divisor and proceed as in multiplication.*

EXAMPLES.

(1) Find the value of the compound fraction $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$.

$$\text{Product} = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}.$$

(Cancelling, *i.e.*, dividing numerator and denominator by the product of the common factors 2, 3. Art. 56.)

$$\text{Product} = \frac{1}{4}.$$

(2) Multiply $\frac{8}{9}, \frac{16}{24}, \frac{27}{30}, \frac{45}{60}$ together.

$$\text{Product} = \frac{8 \times 16 \times 27 \times 45}{9 \times 24 \times 30 \times 60}$$

$$= \frac{8 \times 4 \times 4 \times 3 \times 9 \times 5 \times 9}{9 \times 3 \times 8 \times 5 \times 6 \times 5 \times 12}.$$

(Cancelling, *i. e.*, dividing the numerator and denominator by the product of the common factors 8, 3, 9, 5.)

$$\text{Product} = \frac{4 \times 4 \times 9}{6 \times 5 \times 12} = \frac{4 \times 2 \times 2 \times 3 \times 3}{3 \times 2 \times 5 \times 3 \times 4}.$$

Cancelling again, we get the product in its lowest terms $= \frac{2}{5}$.

(3) Divide $\frac{3}{4}$ of $\frac{7}{8}$ by $\frac{15}{16}$ of 7.

$$\frac{3}{4} \text{ of } \frac{7}{8} \div \frac{15}{16} \text{ of } 7 = \frac{3 \times 7}{4 \times 8} \div \frac{15 \times 7}{16 \times 1} = \frac{3 \times 7}{4 \times 8} \times \frac{16 \times 1}{15 \times 7}$$

$$= \frac{3 \times 7 \times 16}{4 \times 8 \times 15 \times 7} = \frac{3 \times 7 \times 4 \times 4}{4 \times 2 \times 4 \times 3 \times 5 \times 7} = \frac{1}{10}.$$

82. *On Exponents.* It was shown (Art. 7) that $x\,x\,x = x^3$ and $x\,x\,x\,x = x^4$ $\therefore x^3 \times x^4 = x\,x\,x\,x\,x\,x\,x = x^7$, the index of the product being the sum of the indices of the factors,

$$\therefore x^3 \times x^4 = x^{3+4} = x^7 \ldots\ldots\ldots\ldots\ldots\ldots (\alpha)$$

For fractional exponents:

$$\left.\begin{aligned} x^{\frac{1}{2}} \times x^{\frac{1}{2}} &= x^{\frac{1}{2}+\frac{1}{2}} \\ x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} &= x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \end{aligned}\right\} = x.$$

And, therefore, $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, are used to express the square and cube roots of x; consequently,

$$x \times x^{\frac{1}{2}} = x^{\frac{2}{2}} \times x^{\frac{1}{2}} = x^{\frac{2}{2}} + {}^{\frac{1}{2}} = x^{\frac{3}{2}} \ldots\ldots\ldots (\gamma)$$

Similarly by the rules for the division of fractions:

$$\frac{x^6}{x^4} = \frac{x^{4+2}}{x^4} = x^2;\ \text{also shown,} \frac{x^6}{x^4} = x^{6-4} = x^2 \ldots\ldots (\epsilon)$$

In this way negative indices are obtained, and similarly treated; thus—

$$\frac{x^3}{x^4} = \frac{x^3}{x^{3+1}} = \frac{1}{x};\ \text{otherwise,} \frac{x^3}{x^4} = x^{3-4} = x^{-1}.$$

$$\therefore x^{-1} \times x^{-3} = \frac{1}{x} \times \frac{1}{x^3};\quad ,,\quad x^{-(1+3)} = x^{-4} \ldots\ldots\ldots (\beta)$$

Again, for division of negative and fractional exponents;

$$\frac{x^{-3}}{x^{-2}} = \frac{1}{x^3} \div \frac{1}{x^2} = \frac{x^2}{x^3} = x^{2-3} = x^{-1};$$

$$\text{or,} \frac{x^3}{x^2} = x^{-3-(-2)} = x^{-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\zeta)$$

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Finally, $\frac{x}{x^{\frac{1}{2}}} = x^{\frac{2}{2}-\frac{1}{2}} = x^{\frac{1}{2}}$ . . . . (θ)

(1) Hence, to multiply powers, (*positive* (α), *negative* (β), or *fractional* (γ),) the rule is to add their exponents.

(2) To divide one power (*positive* (ϵ), *negative* (ζ), or *fractional* (θ),) by another, the rule is to subtract the exponent of the divisor from that of the dividend.

(3) Again $\frac{x^3}{x^3} = 1$, otherwise $\frac{x^3}{x^3} = x^{3-3} = x^0$ *i.e.* $1 = x^0$, which means that any quantity whatever, whose exponent is 0, is unity.

(4) The general expression for fractional exponents, is $x^{\frac{m}{n}}$; which is read, the *n*th root of x to the *m*th power.

DECIMAL FRACTIONS.

83. To add or subtract fractions, it is necessary (Art. 65) that they should have a common denominator; and the fractions with 10 or a power of 10 in their denominator, called decimal fractions, are more readily reduced to a common denominator, and can be used with much greater facility than any other fractions.

In whole numbers the decimal notation confers a local value on each figure, as has been explained, and any one figure being removed one place to the right has one-tenth of its former value.

In decimal fractions this notation is continued to the right of the unit's place, to mark which, a point, called a decimal point, is placed on the right of the unit's figure.

Thus in the fraction $\frac{48}{100}$, the denominator is dispensed with, and it is written ·48.

$$\text{Since } \cdot 48 = \frac{48}{100}$$

$$\cdot 048 = \frac{48}{1000}$$

$$\cdot 480 = \frac{480}{1000} = \frac{48}{100}$$

we see that ·48, ·048, ·480 are respectively equivalent to fractions which have the same numerator, and the first and third of which have also the same denominator, while the denominator of the second is greater. Consequently ·48 is equal to ·480, but ·048 is less than either. (Art. 62.)

The value of a decimal is therefore not affected by *affixing* ciphers to the right of it; but its value is decreased by *prefixing* ciphers; which effect is exactly opposite to that which is produced by affixing and prefixing ciphers to integers.

ADDITION OF DECIMALS.

84. Rule. Place the numbers under each other, units under units, tens under tens, &c., one-tenths under one-tenths, &c., so that the decimals be all under each other: add, as in whole numbers, and place the decimal point in the sum under the decimal point above.

Ex. Add together 127·5037, ·042, 342, and 2·15.

$$\begin{array}{r@{}l} 127 & \cdot5037 \\ & \cdot042 \\ 342 & \cdot \\ 2 & \cdot15 \\ \hline 471 & \cdot6957 \end{array}$$

NOTE.—The same method of explanation holds for the fundamental rules of decimals, which has been given at length in explaining the rules for Simple Addition, Simple Subtraction, and the other fundamental rules in whole numbers.

Reason for the above Process.—If we convert the decimals into fractions, and add them together as such, we get—

$$127{\cdot}5037 + {\cdot}042 + 342 + 2{\cdot}15$$
$$= \frac{1275037}{10000} + \frac{42}{1000} + \frac{342}{1} + \frac{215}{100}.$$

(Or reducing to a common denominator)—

$$= \frac{1275037}{10000} + \frac{420}{10000} + \frac{3420000}{10000} + \frac{21500}{10000}$$
$$= \frac{4716957}{10000}$$
$$= 471{\cdot}6957$$

SUBTRACTION OF DECIMALS.

Rule. Place the less number under the greater, so that the decimal points may be exactly under each other; suppose ciphers to be supplied, if necessary, in the upper line to the right of the decimal, then proceed as in Simple Subtraction of Integers, and place the decimal point under the decimal point above.

Ex. Subtract 5·4216 from 9·825.

$$\begin{array}{r@{}l} 9 & \cdot825 \\ 5 & \cdot4216 \\ \hline 4 & \cdot4034 \end{array}$$

Reason for the above Process.—If we convert the decimals into fractions, and subtract the one from the other as such, we get—

$$\begin{aligned}9{\cdot}825 - 5{\cdot}4216 &= \frac{9825}{1000} - \frac{54216}{10000}\\ &= \frac{98250}{10000} - \frac{54216}{10000}\\ &= \frac{44034}{10000}\\ &= 4{\cdot}4034\end{aligned}$$

85. To reduce a vulgar to a decimal fraction, for example $\frac{36}{75}$, multiply the numerator and denominator successively by 10, 100, 1000, &c. If, on continuing this multiplication, which does not alter the value of the fraction (Art. 55), the numerator is ever divisible by 75, the fraction can be reduced to a decimal fraction, by dividing the numerator thus found and the corresponding denominator by 75, which will not alter the value of the fraction. (Art. 56).

In $\frac{36}{75}$, after multiplying numerator and denominator by 100, which gives $\frac{3600}{7500}$, the numerator is divisible by 75, and the fraction becomes $\frac{48}{100}$ written ·48.

$$\begin{array}{r}75)360(48\\ 300\\ \hline 600\\ 600\\ \hline\end{array}$$

86. The usual practice in reducing a vulgar to a decimal fraction is as above, to divide numerator by denominator, adding ciphers until there is no remainder, and then to count the number of ciphers employed, which gives the number of figures that must be marked off in the quotient, *from the right*, by the decimal point, prefixing ciphers if necessary to make up the number.

87. It will generally happen that the annexing of ciphers to the numerator will never make it divisible by the denominator. Thus, in reducing $\frac{6}{7}$ and $\frac{1}{9}$ to decimals, the quotients obtained are a continued recurrence of the same figures ·8571428571428, &c., and ·111, &c. All such fractions are called *recurring* or *circulating* decimals, and the figures (in the first case 6) which are thus repeated are called a period. Instead of repeating the recurring figures, a point is placed over the first and last figure of the period, and over the recurring figure if a single digit, thus, $\cdot\dot{8}5714\dot{2}$, $\cdot\dot{1}$.

88. To account for the recurrence of the same figures in reducing a vulgar to a decimal fraction, it should be kept in mind that every remainder must be less than the divisor, and therefore, in every case, the greatest possible variety of remainders will always be one less than the divisor. Consequently the division must terminate without a remainder, or, after obtaining a certain number of remainders, (*i. e.*, one less than the number of the divisor), one of the previous remainders will recur, and then the figures in the quotient will be repeated. Examination of any one instance will make this evident.

No vulgar or common fraction can be reduced to a decimal that terminates, except when, being in its lowest terms, its denominator contains only the powers of 2 and 5 as factors; because 10 contains only these two factors, and in order to reduce a vulgar fraction to a decimal we have to multiply the numerator by some power of 10, and then divide by the denominator; when therefore a fraction is in its lowest terms, and its denominator does not measure 10 or any power of 10, it cannot measure the numerator multiplied by 10 or any power of 10. When therefore a fraction is reduced to its lowest terms, it depends entirely on the denominator whether or not it can be reduced to a decimal exactly.

Expressed generally, a terminating decimal in its lowest terms must be of the form $\frac{a}{2^p \cdot 5^q}$, *and the number of decimal places is p or q, as p is greater or less than q.* To prove this:

1. Let p be greater than q, $\frac{a}{2^p\, 5^q} = \frac{a \cdot 5^{-q}}{2^p} = \frac{a \cdot 5^{-q} \cdot 5^p}{2^p \cdot 5^p} = \frac{a \cdot 5^{p-q}}{10^p}$, which contains p decimals.

2. Let p be less than q, $\frac{a}{2^p \cdot 5^q} = \frac{a \cdot 2^{-p}}{5^q} = \frac{a \cdot 2^{-p} \cdot 2^q}{5^q \cdot 2^q} = \frac{a \cdot 2^{q-p}}{10^q}$, which contains q decimals.

NOTE.—1°. Any prime number p, in the denominator, forms a period of $(p-1)$ places, or a factor of this number; thus $\frac{5}{7} = \cdot\dot{7}1428\dot{5}$ and $\frac{3}{13} = \cdot\dot{2}3076\dot{9}$.

2°. When the period $(p-1)$ is multiplied by any number under p, the product contains the same digits in a different order; thus $\frac{1}{7}, \frac{2}{7}, \frac{3}{7} \ldots \frac{6}{7}$, all have the same figures in their quotients, only in a different order.

3°. When the period $(p-1)$ is very long, it is useful to know that the latter half of it may be obtained from the former, by taking each figure in succession from 9. The period obtained from $\frac{1}{29}$ shows this:

Former half, ·03448275862068 (by actual division),
Remaining do., 96551724137931 (by subtraction from 9),
and $\frac{1}{29} = \cdot\dot{0}344827586206896551724137931$.

89. 1. *To reduce a finite decimal to a vulgar fraction:* Taking the given decimal as a numerator, write for a denominator 1, followed by as many ciphers as there are figures in the given decimal, and make the necessary reduction by cancelling. Thus,

$$\cdot 125 = \frac{125}{1000} = \frac{1}{8},\ \cdot 025 = \frac{25}{1000} = \frac{1}{40}.$$

When a given decimal recurs or circulates (Art. 87), if the period begins at the decimal point, it is called a *pure* circulator, all others are called *impure*, or *mixed.*

2. *To reduce a pure circulator to a vulgar fraction.* Take a complete period for the numerator, and write for the denominator as many 9s as there are figures in the period. Thus,

$$\cdot 3636\dot{3}\dot{6} = \frac{36}{99} = \frac{4}{11}.$$

Proof.

$$\text{Let}\quad x = \cdot 36\dot{3}\dot{6}$$
$$\therefore 100\,x = 36\cdot\dot{3}\dot{6}$$
$$\text{Subtracting, } 99\,x = 36$$
$$x = \frac{36}{99} = \frac{4}{11}.$$

3. *To reduce an impure circulator to a vulgar fraction.* Subtract the part that does not recur from the complete decimal; the remainder forms the numerator, the denominator to which consists of as many 9s as there are recurring figures, followed by as many ciphers as there are figures which do not recur.

Thus, for instance, (1) $\cdot 24\dot{5}\dot{7}$ and (2) $\cdot 5725446\dot{4}2857\dot{1}$.

(1)

$$\begin{array}{lr} \text{From} & 2457 \\ \text{Take} & 24 \\ \hline & 2433 = \text{Numerator.} \end{array}$$

Two figures do not recur, and two do,

$\therefore$ denom. = 9900

$$\text{And}\quad 24\dot{5}\dot{7} = \frac{2433}{9900} = \frac{811}{3300}.$$

(2)

$$\begin{array}{lr} \text{From} & 5725446428571 \\ \text{Take} & 5725446 \\ \hline & 5725440703125 = \text{Numerator.} \end{array}$$

Six figures recur, seven do not; *g. c. m.* = 11160703125.

$$\therefore\ \cdot 5725446\dot{4}2857\dot{1} = \frac{5725440703125}{9999990000000} = \frac{513}{896}.$$

Proof of the Rule.

$$\text{Let } y = \cdot 24\dot{5}\dot{7}$$
$$\therefore 100\,y = 24\cdot\dot{5}\dot{7}$$
$$10000\,y = 2457\cdot\dot{5}\dot{7}$$

Subtracting the second line from the third we get

$$9900\,y = 2433$$
$$y = \frac{2433}{9900},$$

which, put in the following form,

$y = \frac{2433}{9900} = \frac{2457 - 24}{9900}$, furnishes the rule.

MULTIPLICATION OF DECIMALS.

90. Rule. Multiply as in whole numbers, and point off in the product as many decimal places as there are in both the multiplier and multiplicand; if there are not figures enough, supply the deficiency by prefixing ciphers.

Ex. 1. Multiply 36·125 by 2·35.

$$\begin{array}{r} 36\cdot125 \\ 2\cdot35 \\ \hline 180625 \\ 108375 \\ 72250 \\ \hline 84\cdot89375 \end{array}$$

Here there are five decimal places, *viz*, three in the multiplicand and two in the multiplier; therefore we point off five decimals from the right of the product.

Ex. 2. Multiply ·064 by ·0085.

$$\begin{array}{r} \cdot064 \\ \cdot0085 \\ \hline 320 \\ 512 \\ \hline \cdot0005440 \end{array}$$

Here there are seven decimal figures in the two factors, and having only four figures in the product, we prefix three ciphers to make up the deficiency.

91. *Proof by Vulgar Fractions.*

$$\begin{aligned} 36{\cdot}125 \times 2{\cdot}35 &= 36\frac{125}{1000} \times 2\frac{35}{100} \\ &= \frac{36125}{1000} \times \frac{235}{100} \\ &= \frac{8489375}{100000} = 84\frac{89375}{100000} \\ &= 84{\cdot}89375 \end{aligned}$$

Generally—

Let a (the multiplicand) contain m decimals,
And b (the multiplier) „ n „

These quantities expressed fractionally, will then be $\frac{a}{10^m}$ and $\frac{b}{10^n}$.

∴ the product will be $\frac{ab}{10^{m+n}}$, that is, it will contain $(m + n)$ decimals.

DIVISION OF DECIMALS.

92. Divide exactly as in integers, supplying the dividend with ciphers when required.

If the number of decimal places in the dividend exceed that in the divisor, mark off in the quotient as many decimal places as make up the difference, prefixing ciphers when necessary.

If the number in the divisor and dividend be equal, the quotient is a whole number.

If the number in the divisor exceed that in the dividend, annex as many ciphers to the quotient as make up the difference.

Often in the process we must add ciphers to the remainders, and of course these must be counted as so many ciphers added to the dividend.

Examples. Divide ·00169, ·169, and 169·, by ·013.

·013)·00169(·13	·013)·169(13·	·013)169·(13000·
13	13	13
39	39	39
39	39	39

93. *Proof by Vulgar Fractions.*

$$\frac{169}{100000} \div \frac{13}{1000} = \frac{169}{100000} \times \frac{1000}{13} = \frac{169000}{1300000} = \frac{13000}{100000} = {\cdot}13$$

$$\frac{169}{1000} \div \frac{13}{1000} = \frac{169}{1000} \times \frac{1000}{13} = \frac{169000}{13000} = 13{\cdot}$$

$$\frac{169}{1} \div \frac{13}{1000} = \frac{169}{1} \times \frac{1000}{13} = \frac{169000}{13} = 13000{\cdot}$$

General Proof.

Let (a) the dividend contain (m) decimals,
(b) the divisor „ (n) „

These quantities expressed fractionally will be $\frac{a}{10^m}$ and $\frac{b}{10^n}$, and the quotient will be $\frac{a}{10^m} \times \frac{10^n}{b} = \frac{a}{b} \times \frac{10^n}{10^m}$.

Now if (m) be greater than (n), this quotient will be

$$\frac{a}{b} \times \frac{1}{10^{m-n}} = \frac{\frac{a}{b}}{10^{m-n}},$$

which contains ($m - n$) decimals.

If $m = n$, this becomes $\frac{\frac{a}{b}}{10^0} = \frac{\frac{a}{b}}{1} = \frac{a}{b}$, which contains no decimals.

If n be greater than m, the quotient will be $\frac{a}{b} \times 10^{n-m}$; that is, in this case ($n - m$) ciphers must be added to the quotient.

EXAMPLES.

(1) Reduce $\frac{4015}{6305}$ to a decimal. (*M.* 1838.)

G. c. m. = 5; ∴ lowest terms are $\frac{803}{1261}$, and the decimal

$$= \cdot\dot{6}36796193497224425059476605868358445678033306899$$
$$286280729579698651863600317208564631245043616177\dot{7}$$

(2) Reduce $\frac{113}{625}$ to a decimal, and point out why the decimal terminates. (*M.* 1854.)

$$\frac{113}{625} = \frac{113}{2^0 \times 5^4} = \cdot 1808.$$

(3) Write down all possible denominators of vulgar fractions in their lowest terms, which shall produce finite decimals of exactly *four* places. (*M.* 1854 and 1856.)

The denominators that produce finite decimals of exactly *four* places, are (Art. 88) $2^4.5^x$ and $2^x.5^4$, where x may have any of the values 0, 1, 2, 3, 4.

By substituting successively these values of x,

$2^4.5^x$ gives 16, 80, 400, 2000, 10000; and
$2^x.5^4$ „ 625, 1250, 2500, 5000, 10000;

which are all the possible denominators.

(4) Reduce $\frac{13}{16}$ to a decimal, and show why every vulgar fraction in its lowest terms, with 16 for its denominator, must be equivalent to a finite decimal of four places.

$$\frac{13}{16} = \cdot 8125\text{; and from above } 16 = 2^4.5^0\text{;}$$

∴ the denominator must (Art. 88) contain four decimal places.

(5) Reduce $\frac{(2\cdot 05)^2 \times 2{\cdot}{\cdot}24}{\cdot 0041}$ (*M.* 1860).

$$\frac{(2\cdot 05)^2 \times 2\cdot 24}{\cdot 0041} = \frac{9\cdot 4136}{\cdot 0041} = 2296.$$

(6) Reduce $\frac{1}{181}$ to a decimal.

One-half of the answer is here given, the remaining half can easily be obtained by subtraction from 9, as pointed out in the note to Art. 88.

·005524861878453038674033149171270718232044198,
895027624309392265193370165745856353591160220, &c.

CONCRETE NUMBERS.

94. Hitherto we have considered abstract numbers only, or concrete numbers of one denomination. It is evident that if concrete numbers were all of one denomination,—if, for instance, pounds were the only units of money, miles of length, years of time, and so on,—such numbers would be subject to the common rules for abstract numbers. Or if the concrete numbers were of different denominations, and those denominations differed from each other by 10, or multiples of 10, then all operations with such concrete numbers could be carried on by the rules which have been given for decimals. But generally, with concrete numbers, such a relation does not hold between the different denominations; but the rules already given are easily extended to *concrete* magnitudes wherein the local values of the different figures are connected by more numbers than one, as, for instance, to *pounds*, *shillings*, *pence*, and *farthings*, where the numbers 4, 12 and 20 connect the different denominations, in precisely the same manner as the number 10 was supposed to connect the successive denominations of integers.

NOTE.—Reduction is the changing of numbers from one name or denomination to another without altering their value.

When the numbers are to be reduced from a higher name to a lower, it is called Reduction Descending; when from a lower name to a higher, Reduction Ascending.

In the solution of the following examples, the reader's familiar acquaintance with the usual Tables of Money, Weights and Measures, is assumed, and also with some of the ordinary processes of reduction.

EXAMPLES

In concrete quantities illustrating the preceding rules.

(1) What is the area of a rectangular court of which the length is 250 yards, 1 foot, 6 inches, and the width 32 yards, 2 inches?

Area = 250 yds. 1 ft. 6 in. × 32 yds. 0 ft. 2 in. (*M.* 1838.)

$$= 250\tfrac{1}{2} \times 32\tfrac{1}{18} = \frac{501}{2} \times \frac{577}{18} = \frac{96359}{12} = 8029\tfrac{11}{12} \text{ yards}$$

= 8029 yds. 8 ft. 36 in. *Ans.*

(2) Add together $\frac{5}{8}$ of £1000 16*s.* 8*d.*, $\frac{3}{4}$ of £2400 12*s.* 4*d.*, $\frac{5}{7}$ of £3724 14*s.* 7*d.* (*M.* 1843.)

$$\begin{array}{lllrrr} & & & \pounds & s. & d. \\ \dfrac{5 \times (1000 \,..\, 16 \,..\, 8)}{8} = & \dfrac{5004 \,..\, 3 \,..\, 4}{8} = & & 625 & 10 & 5 \\ \dfrac{3 \times (2400 \,..\, 12 \,..\, 4)}{4} = & \dfrac{7201 \,..\, 17 \,..\, 0}{4} = & & 1800 & 9 & 3 \\ \dfrac{5 \times (3724 \,..\, 14 \,..\, 7)}{7} = & \dfrac{18623 \,..\, 12 \,..\, 11}{7} = & & 2660 & 10 & 5 \\ \hline & & & 5086 & 10 & 1 \end{array} \quad \textit{Ans.}$$

(3) Which is the greatest, $\frac{1}{19}$ of a pound sterling, or $\frac{1}{20}$ of a guinea? Express the difference between them. (*M.* 1844.)

The difference between £1 × $\frac{1}{19}$ and guinea 1 × $\frac{1}{20}$ is

$$\frac{20s.}{19} - \frac{21s.}{20} = \frac{400 - 399}{380} = \frac{1s.}{380} = \pounds\frac{1}{7600}.$$

(4) What fraction of £100 is £3 17*s.* 6*d.*?

$$\frac{\pounds 3\ 17s.\ 6d.}{\pounds 100} = \frac{3\tfrac{7}{8}}{100} = \frac{31}{800}. \quad \textit{Ans.}$$

(5) What fraction of a pound sterling is 17*s.* 6½*d.*? Express in seconds $\frac{7}{13}$ of a year, taking the length of the year as 365 days 6 hours. (*M.* 1845.)

$$\frac{7s.\ 6\tfrac{1}{2}d.}{20s.} = \pounds\frac{181}{480};$$

$$\frac{7 \times 365\ d.\ 6\ h.}{13} = \frac{7(365 \times 24 + 6) \times 60 \times 60}{13} = 16992553\tfrac{11}{13} \text{ secs.}$$

(6) What length of carpet $\frac{3}{4}$ wide will cover a rectangular room 36 feet long, and 27 feet 9 inches wide, and what will be the cost of the carpet at 4*s.* 9*d.* a yard? (*M.* 1848.)

Area = 12 yds. × 9¼ yds.;

$$\text{required length} = \frac{12 \times 9\tfrac{1}{4}}{\tfrac{3}{4}} = \frac{12}{1} \times \frac{37}{4} \times \frac{4}{3} = 148 \text{ yds.}$$

$$148 \times 4\tfrac{3}{4}s. = \frac{148 \times 19}{4} = 37 \times 19 = 703s. = \pounds 35\ 3s. \text{ cost.}$$

(7) Reduce 167805 ounces avoirdupois to tons, &c.

In reducing ounces to pounds, and pounds to quarters by steps of short division, explain the true values of the final remainders in each line of the process. (*M.* 1854.)

$$\begin{array}{lrll}
 & 4\,)\,\underline{167805} & & \\
16 & & & \\
 & 4\,)\,\underline{41951} + 1 & = 1\text{ oz.} & \\
 & & & \therefore 12 + 1 = 13\text{ oz.} \\
 & 4\,)\,\underline{10487} + 3 \times 4 & = 12\text{ oz.} & \\
28 & & & \\
 & 7\,)\,\underline{2621} + 3 & = 3\text{ lb.} & \\
 & & & \therefore 12 + 3 = 15\text{ lb.} \\
 & 4\,)\,\underline{374 + 3 \times 4} & \underline{= 12\text{ lb.}} & \\
 & 2{,}0\,)\,\underline{9{,}3} + 2\text{ qrs.} & & \\
 & 4 \quad + 13\text{ cwt.} & &
\end{array}$$

	Tons.	cwt.	qrs.	lbs.	oz.
∴ *Ans.*	4	13	2	15	13

(8) Express $\frac{7}{9}$ of half-a-guinea as the fraction of a crown. (2) What fraction of 12*s.* 6*d.* must be added to $\frac{5}{7}$ of a guinea to make a pound sterling? (*M.* 1854.)

$$\frac{\frac{7}{9} \times 10\frac{1}{2}}{5} = \frac{7}{9} \times \frac{21}{2} \times \frac{1}{5} = \frac{7 \times 7 \times 3}{3 \times 3 \times 2 \times 5} = \frac{7 \times 7}{3 \times 2 \times 5} = \frac{49}{30}.$$

(2) $20s. - \frac{5}{7} \times 21s. = 20 - 15 = 5s.\ \therefore \frac{5}{12\frac{1}{2}} = \frac{5}{\frac{25}{2}} = \frac{5}{1} \times \frac{2}{25} = \frac{2}{5}$ *Ans.*

(9) How many half-crowns are there in £756 17*s.* 6*d.*? (2) How many years are there in 7305 days, the length of the year being taken at $365\frac{1}{4}$ days? In the last example explain briefly the process employed. (*M.* 1855.)

$$£756\ 17s.\ 6d. = 756\tfrac{7}{8} = \frac{6055}{8};\ \therefore \textit{Ans.} = 6055.$$

$$(2)\quad \frac{7305}{365\frac{1}{4}} = \frac{29220}{1461} = 20 \text{ years.}$$

Here the numerator and denominator are each multiplied by 4. The resulting numerator is then divided by its denominator.

(10) If 22 yards = 1 chain and 4840 square yards = 1 acre, how many square chains are there in 5 acres? (*M.* 1857.)

22 yds. = 1 chain.
22 × 22 = 484 sq. yds. = 1 sq. chain.
∴ 4840 (or 10 × 484) sq. yds. = 10 sq. chains = 1 acre.
Hence 5 × 10 = 50 sq. chains = 5 acres.

(11) The value of an ounce of standard gold is £3 17*s.* $10\frac{1}{2}$*d.*; what fraction of one million sterling are 625 ounces of gold? (*M.* 1859.)

By decimals, 12) 10·5 = 10½*d.*

20) 17·875 = 17*s.* 10½*d.*

3·89375 = £3 17*s.* 10½*d.*

$$\frac{3{\cdot}89375 \times 625}{1{,}000{,}000} = \frac{2433{\cdot}59375}{1{,}000{,}000} = {\cdot}00243359375.$$

By vulgar fractions,

$$\frac{625\{[(3 \times 20 + 17) \times 12 + 10] \times 2 + 1\}}{1{,}000{,}000 \times 20 \times 12 \times 2} = \frac{1168125}{480{,}000{,}000}$$

$$= \frac{623}{256000} = {\cdot}00243359375, \text{ as before.}$$

(12) Prove that 17·975 of £71 2*s.* is equal to $\frac{1}{40}$ of £51120 18*s.* (*M.* 1861.)

$$\left.\begin{array}{l} £71\ 2s. = 71{\cdot}1, \text{ and} \\ £51120\ 18s. = 51120{\cdot}9; \end{array}\right\} \left.\begin{array}{c} 71{\cdot}1 \times 17{\cdot}975 \\ \dfrac{51120{\cdot}9}{40} \end{array}\right\} = 1278{\cdot}0225.$$

(13) If 25 tons of goods are purchased for £37 10*s.*, and sold at 35*s.* a ton, what is the gain per ton? (2) At what rate should the goods have been sold in order to obtain a profit of £9 7*s.* 6*d.*? (*M.* 1862.)

$$\left.\begin{array}{r} £37\ 10s.\ 0d. = 37{\cdot}5 \\ £1\ 15s.\ 0d. = 1{\cdot}75 \\ £9\ \ 7s.\ 6d. = 9{\cdot}375 \end{array}\right\} \frac{(1{\cdot}75 \times 25) - 37{\cdot}5}{25} = {\cdot}25 = 5s. \text{ gain per ton.}$$

$$(2)\quad \frac{37{\cdot}5 + 9{\cdot}375}{25} = 1{\cdot}875 = £1\ 17s.\ 6d. \text{ rate per ton.}$$

(14) What fraction of the earth's diameter (7900 miles) is a mountain 4½ miles high? By what fraction of an inch would the height of such a mountain be properly represented on a globe of 18 inches diameter? (*M.* 1849.)

$$(1)\ \frac{4\frac{1}{2}}{7900} = \frac{9}{15800}; \text{ and } (2)\ \frac{9}{15800} \times 18 = \frac{81}{7900} \text{ of an inch.}$$

(15) At what time next before 12 o'clock are the hour and minute hands of a watch together? (*M.* 1838.)

The conjunctions from XII. to XII. again are eleven in all, the eleventh taking place at the end of the twelve hours, and since the hands move equably, these conjunctions must happen at eleven points equidistant from each other. Therefore, if the circumference be divided into eleven equal parts, we shall have the conjunction that happens between

I. and II. at $\frac{1}{11}$ of 60 minutes past I.
II. and III. at $\frac{2}{11}$ of „ past II.

. . . .

X. and XI. at $\frac{10}{11}$ of „ past X.
that is at 54 minutes $32\frac{8}{11}$ seconds past X.
the time required.

CHAPTER III.

SQUARE ROOT.

95. THE square root of any number or quantity is such a number or quantity as being multiplied by itself produces it. Thus the square root of 49 (written $\sqrt{49}$ or $49^{\frac{1}{2}}$) is 7, because $7 \times 7 = 49$.

Since the square roots of 1, 4, 9, 16, 25, 36, 49, 64, 81,
are 1, 2, 3, 4, 5, 6, 7, 8, 9,
we can, by mere inspection, find the square roots of all quantities that are produced by the squaring of a single figure.

Also, since the square root of 1 is 1·
———— 100 „ 10·
———— 10000 „ 100·
———— 1000000 „ 1000· &c.,

it is easily seen that the square root of a number of fewer than three figures must consist of only one figure; that of a number of more than two figures, and fewer than five, of two figures; that of a number of more than four figures, and fewer than seven, of three figures; and so on; whence it follows, that if a point be placed over every alternate figure, beginning at the unit's place, the number of such points will be the same as the number of figures in the square root.

This is called the *Rule for Pointing*, and may easily be extended to decimals; thus—

Since the square root of ·01 is ·1:
———— ·0001 „ ·01:
———— ·000001 „ ·001: &c.,

we infer that the quantity proposed must first be made to have an *even* number of decimal places, and then that the pointing must be made from the place of units towards the right hand over every alternate figure as before.

The rule for extracting the square root is derived from the formula

$$(a + b)^2 = a^2 + (2\,a + b)\,b.$$

Ex. 1. Extract the square root of 1679616.

96. Pointing the unit's place and every second figure, and taking the highest square from the first period, also putting the root in the quotient, we have

$$\begin{array}{l} \dot{1}6\dot{7}9\dot{6}1\dot{6}\ (1000 \\ \underline{1000000} \\ 679616 \end{array}$$

$$(a + b)^2 = a^2 + b\,(2\,a + b).$$

The number 1679616 may be separated into two parts, represented algebraically by the right side of the above equation:

$$1000000 + 679616 = a^2 + b\,(2\,a + b).$$

We know from the formula, that the root of $a^2 + b\,(2\,a + b)$ is $a + b$, and a, the root of the first period, is found by inspection, while b, the root of the remainder, has to be obtained by the formula; thus—

$$\text{The remainder, or } b\,(2\,a + b) = 679616,$$

$$\therefore b = \frac{679616}{2a + b}.$$

As in the denominator $2\,a + b$, the quantity b is always small compared with $2\,a$, the number of times that $2a$, or twice the root already found, is contained in the remainder, generally gives the correct number for the root of the remainder.

97. If, as will often be the case, $b\,(2\,a + b)$ does not exactly equal the remainder, there will be a second remainder; then the whole of the root (the two parts) found, equals a, and the same process is repeated to find the root of the second remainder, and so on until the last period is brought down: when, if there be no remainder, the exact root is found, and, if there be a remainder, the number is not an exact square. The above operations are thus represented:

$$\begin{array}{rrl} 1000\,)\!\! & 1679616 & (\,1000 \\ & \underline{1000000} & 200 \\ & & 90 \\ & 679616 & \underline{6} \\ 200\,(2 \times 1000 + 200) = & \underline{440000} & 1296 \\ & 239616 & \\ 90\,(2 \times 1200 + 90) = & \underline{224100} & \\ & 15516 & \\ 6\,(2 \times 1290 + 6) = & \underline{15516} & \end{array}$$

For convenience, in practice, the arrangement of the figures differs from the preceding; repetition is avoided by bringing down one period at a time, and the divisor is formed by multiplying the root by two and leaving the unit's place, in which is put the root of the remainder.

$\dot{1}6\dot{7}9\dot{6}1\dot{6}$ (1296
1

22) 67
44

249) 2396
2241

2586) 15516
15516

The consideration of what precedes furnishes the following

98. Rule. Place a point over the *unit's* figure, and over every alternate figure right and left, thus forming as many *periods* of two figures each as possible. Find the greatest square number contained in the first period on the left hand, and subtract it from that period, putting down its root on the right, as in division.

To the remainder bring down the next period for a dividend: double the root just found for a divisor, and find how often it is contained in this dividend exclusive of the figure on its right hand; annex this quotient to the figures in both the quotient and divisor; multiply the divisor thus formed by the last figure of the quotient, subtract the product, and proceed as before.

Repeat this process till every period is disposed of, and the root, or an approximation to it will thus be obtained.

Ex. 1. Extract the square root of 9512295961.

$9\dot{5}1\dot{2}2\dot{9}5\dot{9}6\dot{1}$ (97531
81

187) 1412
1309

1945) 10329
9725

19503) 60459
58509

195061) 195061
195061

Ex. 2. Extract the square root of ·0758329. Making the number of decimal places *even* and pointings, we have

$$\begin{array}{r l}
 & 0{\cdot}\dot{0}7\dot{5}8\dot{3}2\dot{9}0\ (\,{\cdot}2753 \\
 & 4 \\ \hline
47\,) & 358 \\
 & 329 \\ \hline
545\,) & 2932 \\
 & 2725 \\ \hline
5503\,) & 20790 \\
 & 16509 \\ \hline
 & 4281
\end{array}$$

The former of these is a complete square, whose root is 97531; the latter is not, its approximate root being ·2753, with a remainder ·00004281: and it will be found upon trial that $({\cdot}2753)^2 + {\cdot}00004281 = {\cdot}0758329$: this approximation might evidently have been carried further by affixing periods of *ciphers*, which do not affect the value of the decimal.

99. (1) The square root of a vulgar fraction is the square root of the numerator divided by the square root of the denominator.

Thus $\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$: but as the numerators and denominators are seldom complete squares, it is usual to express the fraction decimally before the rule is applied.

$$\sqrt{\frac{5}{8}} = \sqrt{{\cdot}625} = {\cdot}7905694 +.$$

(2) A whole number, which is not the square of another whole number, cannot have an exact fractional number for its square root.

To prove this generally, let a be the numerator and b the denominator of a fraction expressed in the lowest terms. Then if $\frac{a}{b}$ is the square root of a whole number, $\frac{a}{b} \times \frac{a}{b}$ or $\frac{a^2}{b^2}$ must be a whole number, but this is impossible; for a^2, b^2, contain no prime factors which are not also contained in a, b; and since a, b, are prime to each other, so are a^2, b^2. Then $\frac{a^2}{b^2}$ being an irreducible fraction, cannot be equal to a whole number. The square root of a whole number which is not the square of another whole number is, consequently, not expressible by any exact number; $\sqrt{5}$, $\sqrt{6}$, $\sqrt{13}$ . . . . are therefore not expressible by any exact number.

(3) A number whose last figure is 2, 3, 7, or 8 cannot be a perfect square.

For the simple units of the square of a number, consisting of several figures, arise from the square of the units of the root. But the square of none of the nine digits ends in 2, 3, 7, or 8.

CHAPTER IV.

ARITHMETICAL PROPORTION.

100. **Ratio** is the relative value which quantities of the same kind bear to each other in respect of magnitude.

Thus 6 is twice as great as 3, and 2 is twice as great as 1; therefore we should say that the ratio of 6 to 3 is the same as that of 2 to 1, or as we may write for shortness' sake, 6 : 3 : : 2 : 1.

In speaking of the ratio of two quantities $a : b$, a and b are called the *terms* of the ratio, and a is distinguished as the *antecedent*, b as the *consequent*.

The ratios of the squares of two quantities are said to be *duplicate* of the ratios of the quantities themselves, and that of their cubes *triplicate;* so the ratio of their square roots are said to be *subduplicate*, and that of their cube roots *subtriplicate*. Thus, the ratio of 81 : 16 is duplicate of the ratio of 9 : 4, and the ratio of 3 : 2 is subduplicate of the ratio of 9 : 4. Also the ratio of $27^3 : 8^3$ is triplicate of the ratio of 27 : 8, and the ratio of 3 : 2 is subtriplicate of the ratio of 27 : 8. The antecedents of two or more ratios are called homologous terms, and so are the consequents.

Ratios are said to be *compounded* when their homologous terms are multiplied together, and the ratio of the two products is said to be *compounded* of the simple ratios, and is hence called a *compound* ratio.

101. An equality of ratios constitutes a proportion; thus $\frac{8}{12} = \frac{2}{3}$; here the four digits are proportional, and are usually read, 8 : 12 : : 2 : 3, where 8 and 3 are called the *extreme*, and 12 2 the *mean* terms.

102. In every proportion, the product of the extremes equals the product of the means; thus, if 8 : 12 : : 2 : 3, then $8 \times 3 = 12 \times 2$; for if $\frac{8}{12} = \frac{2}{3}$ and both fractions be brought to a common denominator and multiplied by it, which does not affect the equality, we obtain $\frac{8 \times 3}{12 \times 3} = \frac{2 \times 12}{3 \times 12}$ or $8 \times 3 = 2 \times 12$. Similarly the converse of this proposition is true, that

103. Whenever the product of any two numbers is equal to the product of other two, the four are proportional when so arranged that the two factors of one product shall be *extremes*,

and the two factors of the other product *means*, or *vice versâ*. In this way eight proportions are obtained.

From $8 \times 3 = 12 \times 2$ arise

$$\begin{array}{ll} 8 : 12 :: 2 : 3 & 2 : 3 :: 8 : 12 \\ 8 : 2 :: 12 : 3 & 12 : 3 :: 8 : 2 \\ 3 : 12 :: 2 : 8 & 2 : 8 :: 3 : 12 \\ 3 : 2 :: 12 : 8 & 12 : 8 :: 2 : 3 \end{array}$$

104. In any of these proportions, three terms being given we can find the fourth. If the two means and an extreme be given to find another extreme, we know that the product of the two means is equal to the product of the extreme given, and the one to be found; if, therefore, we divide the product of the given means by the given extreme, we shall obtain the desired number. For the same reason, to find a mean when the other terms are given, divide the product of the given extremes by the given mean.

105. To apply these rules to the previous example for finding the four terms in succession, the mark of interrogation being used to point out the term required, we have

$$? : 12 :: 2 : 3 \quad \text{here } \frac{12 \times 2}{3} = 8, \text{ the first term.}$$

$$8 : ? :: 2 : 3 \quad \text{here } \frac{8 \times 3}{2} = 12, \text{ the second term.}$$

$$8 : 12 :: ? : 3 \quad \text{here } \frac{8 \times 3}{12} = 2, \text{ the third term.}$$

$$8 : 12 :: 2 : ? \quad \text{here } \frac{12 \times 2}{8} = 3, \text{ the fourth term.}$$

106. Here it may be noted, as important to be kept in mind, that in applying the following rules to concrete numbers, no ratio can exist between quantities *differing in kind;* and in comparing quantities of the *same* kind they must be reduced to the same denominator before we can tell their relative value, or ratio. Thus the ratio of 5*s*. to 1*s*. 8*d*. is that of 60*d*. to 20*d*. $= \frac{60}{20}$ or 3 : 1. The ratio of *any* two quantities is an abstract number; thus the ratios of 8 miles to 56 miles, 3 cwt. to 21 cwt., 5 yards to 35 yards, is the ratio of 1 to 7, or $\frac{1}{7}$ in each case.

107. *Rule of Three.*—Every question in proportion furnishes three terms to find the fourth: two of these terms are always of the same kind, being an *antecedent* and its *consequent* of one of the ratios constituting the proportion. The remaining term is of a different kind from the other two, and of the *same kind*

as the *fourth term*, or *answer* which is required. This remaining term is the antecedent, to which the answer is the consequent of the other ratio constituting the proportion.

108. The rule for the arrangement of the terms:

Place the single term, which is of the same kind as the answer, in the third place; the arrangement of the other two terms depends entirely on the nature of the question. The terms of the question will show whether the answer should be greater or less than the third term, and the greater or less of the two remaining terms must be placed in the second place accordingly.

109. This rule includes questions in *inverse* proportion. The following are cases, the 1st, of direct proportion, where an increase in one term, *the number of guns*, requires a proportionate increase of another, *the number of men killed;* and the 2nd, of inverse proportion, (because the ratio is *inverted*;) where an *increase* of one term, *the number of cows*, demands a *decrease* of another term, the number of days.

(1) If 30 guns in a battle kill 2100 men, how many men would 210 guns kill at the same rate? (*M.* 1847.)

30 : 210 : : 2100
210
21000
4200
3,0)44100,0
14700

Ans. 14700 men.

(2) If 30 cows eat a quantity of fodder in 2100 days, how long would the same suffice for 210 cows at the same rate?

210 : 30 : : 2100
30
21,0)6300,0(300
63

Ans. 300 days.

110. In any proportion, when a number in the first term is a factor of a number in the second or third term, the process may be shortened by *cancelling*, as in multiplying fractions together.

111. As a case of compound proportion or Double rule of Three, we shall take this question.

If 9 people spend £120 in 8 months, how much will 24 persons, living at the same rate, spend in 5 months? This question combines two in simple proportion, as in the statements below; first, how much 24 people would spend in the *same* time, 8 months; and, secondly, how much 24 would spend in 5 months. This second is a case of inverse proportion as explained.

$$\text{I.}\quad \begin{array}{lr} (1)\ \text{As } 9 : 24 :: & 120 \\ & 24 \\ \hline & 9)2880 \\ \hline (2)\quad 8 : 5 :: & 320 \\ & 5 \\ \hline & 8)1600 \\ \hline & 200\ \textit{Ans.} \end{array} \qquad\qquad \text{II.}\quad \begin{array}{l} \text{As } 9 : 24 :: 120 \\ \quad\ \ 8 : 5 \end{array}$$

If, instead of making two separate operations, as in I., we make one statement, as in II., and take $\frac{120 \times 24 \times 5}{9 \times 8} = 200$, the same result can be obtained by one process.

III. The following question is an instructive instance of the advantage of cancelling, and also of the extent to which questions in compound proportion may be carried. Here we have five statements combined into one, and the answer is obtained by cancelling only.

If 12 men dig a canal 50 yards long, 9 yards broad, and 6 feet deep, in 90 days, working 6 hours each day, how many men can dig a canal 40 yards long, 4 yards broad, and 10 feet deep, in 40 days, working 8 hours each day?

$$\left.\begin{array}{rcl} 50 \text{ length} & : & 40 \text{ length} \\ 9 \text{ breadth} & : & 4 \text{ breadth} \\ 6 \text{ depth} & : & 10 \text{ depth} \\ 40 \text{ days} & : & 90 \text{ days} \\ 8 \text{ hours} & : & 6 \text{ hours} \end{array}\right\} :: 12 \text{ men.}$$

$$\frac{40 \times 4 \times 10 \times 90 \times 6 \times 12}{50 \times 9 \times 6 \times 40 \times 8}$$

$$= \frac{40 \times 4 \times 10 \times 9 \times 5 \times 2 \times 6 \times 12}{10 \times 5 \times 9 \times 6 \times 40 \times 4 \times 2}.$$

(Cancelling, *i.e.*, dividing the numerator and denominator by the product of the common factors 10, 5, 9, 6, 40, 4, 2,)

Product = 12. *Ans.*

The cancelling may be done in the first statement; the above repetition of the quantities in a fractional form is quite unnecessary in practice.

EXAMPLES.

⁂ *It is expected that the student will work out in full the statements below, to verify the given answers.*

(1) Define proportion, and prove that if four quantities are proportional, the product of the extremes is equal to that of the means. (*M.* 1840.) Answered in Arts. 101 and 102.

(2) If 7 oxen eat an acre of grass in 6 days, how long will it take 17 oxen to eat 34 acres? (*M.* 1841.)

$$\left.\begin{array}{r}17 \text{ oxen} \\ 1 \text{ acre}\end{array}\right\} : \left.\begin{array}{r}7 \text{ oxen} \\ 34 \text{ acres}\end{array}\right\} :: 6 \text{ days} : 84 \text{ days. } \textit{Ans.}$$

(3) Find a fourth proportional to ·0004, 1·4, ·002. (*M.* 1845.)

·0004 : 1·4 : : ·002 : 7. *Ans.*

(4) Define the terms, *ratio, duplicate ratio, proportion.* When is one quantity said to vary inversely as another? (*M.* 1846.)

Answered in Arts. 100, 101, 109.

(5) If 12 men can perform a piece of work in 8 days, in what time will 48 men perform the same? (*M.* 1856.)

48 men : 12 men : : 8 days : 2 days. *Ans.*

(6) If 6 men can dig 14 yards per day of a trench 3 feet wide and 2 feet deep, how many men will be required to dig 12 yards in a day of a trench 7 feet wide and 6 feet deep? (*M.* 1860.)

$$\left.\begin{array}{rcl}14 \text{ yds.} & : & 12 \text{ yds.} \\ 3 \text{ ft.} & : & 7 \text{ ft.} \\ 2 \text{ ft.} & ; & 6 \text{ ft.}\end{array}\right\} :: 6 \text{ men} : 36 \text{ men. } \textit{Ans.}$$

(7) If the wages of 6 men for 5 weeks be £6, how long will 8 men work for £10? (*M.* 1863.)

$$\left.\begin{array}{rcl}8 \text{ men} & : & 6 \text{ men} \\ £6 & : & £10\end{array}\right\} :: 5 \text{ weeks} : 6\tfrac{1}{4} \text{ weeks. } \textit{Ans.}$$

(8) How many men can complete a trench of 468 yards in 8 days, if 24 men can dig 81 yards in 6 days? (*B. A.* 1841.)

$$\left.\begin{array}{rcl}81 \text{ y.} & : & 468 \text{ y.} \\ 8 \text{ d.} & : & 6 \text{ d.}\end{array}\right\} :: 24 \text{ m.} : 104 \text{ m. } \textit{Ans.}$$

(9) If 180 men, in 6 days, working during 10 hours each day, can dig a trench 200 yards long, 3 wide, and 2 deep; in how many days, working during 8 hours each day, will 100 men dig a trench 360 yards long, 4 wide, and 3 deep? (*B. A.* 1857.)

$$\left.\begin{array}{rcl}100 \text{ m.} & : & 180 \text{ m.} \\ 8 \text{ h.} & : & 10 \text{ h.} \\ 200 \text{ y.} & : & 360 \text{ y.} \\ 3 \text{ w.} & : & 4 \text{ w.} \\ 2 \text{ d.} & : & 3 \text{ d.}\end{array}\right\} :: 6 \text{ days} : 48\tfrac{3}{5} \text{ days. } \textit{Ans.}$$

(10) If 81 bushels of wheat are consumed by 56 men in 5 days, how long will 16 men take to consume 28 bushels? (*B. A.* 1861.)

$$\left.\begin{array}{rcl}81 \text{ b.} & : & 28 \text{ b.} \\ 16 \text{ m.} & : & 56 \text{ m.}\end{array}\right\} :: 5 \text{ days} : 6\tfrac{4}{81} \text{ days. } \textit{Ans.}$$

CHAPTER V.

ARITHMETICAL AND GEOMETRICAL PROGRESSION.

ARITHMETICAL PROGRESSION.

112. An arithmetical progression is a series of numbers which increase or decrease by equal steps; thus the natural numbers, 1, 2, 3, &c., are in arithmetical progression, because each term is one greater than the preceding.

In any arithmetical progression let $S =$ the sum of the series, a the first term, l the last term, d the common difference, and n the number of terms.

113. Since the second, third, fourth, &c., terms of the series, are respectively formed by the addition of the first term a to the common difference d, taken once, twice, three times, &c., therefore the nth, or last term, will be,

$$a + (n - 1)\,d, \text{ or, } l = a + (n - 1)\,d \;\ldots\ldots\; \text{(I.)}$$

114. Taking any series of numbers in arithmetical progression consisting of an odd number of terms, it will be seen, that, by the constitution of the series, the first term, a, is as much less than the middle term as the last term is greater, and therefore twice the middle term must be equal to $(a + l)$ the sum of the first and last terms.

$$\therefore \text{middle term} = \frac{a + l}{2}.$$

It is also evident that, from the law of formation of the series, the sum of *any* two terms equally distant from the middle term, is equal to twice the middle term; therefore, if the middle term be multiplied by the number of terms, n, we obtain the sum,

$$\therefore S = n \times \text{middle term} = n\,\frac{a + l}{2} \;\ldots\ldots\; \text{(II.)}$$

Substituting in II, the values of l and a in I, we have

$$S = \frac{(2\,a + (n - 1)\,d)\,n}{2} \;\ldots\ldots\ldots\ldots\; \text{(III.)}$$

$$\text{And } S = \frac{(2\,l - (n - 1)\,d)\,n}{2} \;\ldots\ldots\ldots\ldots\; \text{(IV.)}$$

115. From these four equations, any three of the letters being given, the other two can be found.

Examples.

(1) Find the sum of 12 terms of the series 19, 27, 35, &c. (*M.* 1847.)

Here $a = 19, d = 8, n = 12$; substituting these values in III,

$$S = \frac{(38 + (12 - 1)\,8)\,12}{2} = 756. \text{ Ans.}$$

(2) Insert six arithmetic means between 1 and 29. (*M.* 1839.)

Here $l = 29, a = 1$, and $n = 8$; substituting these values in I,

$$29 = 1 + (8 - 1)\,d. \quad \therefore d = 4,$$

And the series is, 1, 5, 9, 13, 17, 21, 25, 29. *Ans.*

(3) How many terms of the series 1, 3, 5, &c., will amount to 5041? (*M.* 1840.)

Here $a = 1, d = 2, S = 5041$. Substituting, as before, in III,

$$5041 = \frac{(2 + (n - 1)\,2)\,n}{2} = \frac{2\,n + 2\,n^2 - 2\,n}{2} = n^2.$$

$$\therefore n^2 = 5041, \text{ and } n = 71. \text{ Ans.}$$

(4) The sum of n terms of the progression 1, 3, 5, 7, &c., that is of the first n odd numbers, is n^2. (*M.* 1838, *B. A.* 1849, and again *M.* 1855.)

$$\text{Here } S = \frac{\{2\,a + (n - 1)\,d\}\,n}{2} \text{ becomes}$$

$$\frac{\{2 + (n - 1)\,2\}}{2} = \frac{2\,n^2}{2} = n^2.$$

(5) The sum of the first n even numbers is $n^2 + n$.

$$\text{Here } S = \frac{\{4 + (n - 1)\,2\}\,n}{2} = \frac{(2\,n + 2)\,n}{2} = n^2 + n.$$

(6) Insert eleven arithmetic means between 7 and 151. (*M.* 1841.)

The number of terms is 13, *viz.*, 11 means + 2 extremes.

$$\therefore 7 + 12 \times \text{com. diff.} = 151.$$

$$\therefore \text{com. diff.} = 12; \text{ and the series is}$$

7, (19, 31, 43, 55, 67, 79, 91, 103, 115, 127, 139,) 151.

(7) The sum of a decreasing arithmetic series is 140, the first term 10, and the common difference $\frac{1}{3}$; find the number of terms. (*B. A.* 1841.)

$$\text{Here } S = \frac{\{2\,a + (n - 1)\,d\}\,n}{2} \text{ becomes}$$

$$140 = \frac{\left\{20 + (n - 1) \times - \frac{1}{3}\right\}\,n}{2}$$

$$280 = 20\,n - \frac{1}{3}\,n^2 + \frac{1}{3}\,n$$

$$n^2 - 61\,n = -\,840.$$

$$\text{Whence } n = \frac{61}{2} \pm \sqrt{\frac{361}{4}} = \frac{61}{2} \pm \frac{19}{2} = 40 \text{ or } 21.$$

And it will be found, upon trial, that 40 terms of this series give the same sum as 21 terms.

(8) Show how to find the sum of n terms of an arithmetical progression. Find the sums of the following series:—(1) The first fifty even numbers; (2) 19, 14, 9, &c., to 16 terms; (3) $\frac{4}{5}$, 1, $\frac{6}{5}$, &c., to 9 terms. (*M.* 1845.)

Ans. Art. 114 (1) 2550. (2) −296. (3) $14\frac{2}{5}$.

(9) Find the sum of the following series, having investigated the general formulæ on which they depend:—(1) The first ninety odd numbers; (2) The first ninety even numbers; (3) Show that the results in (1) and (2) are included under the forms n^2 and $n^2 + n$; (4) What is meant by the sum of an indefinite series? (*M.* 1846.)

Ans. (1) 8100. (2) 8190. (3) See examples 4 and 5 above. (4) The sum of an indefinite series is the limit to which we continually approach by taking more and more terms of the series, but which cannot be reached by any finite number of terms.

(10) When are magnitudes (1) in arithmetical (2) in geometrical progression? (3) Find an arithmetical and a geometrical mean between a and b; (4) write down the 12th term of the series 7, 12, 17. (*M.* 1849.)

Ans. (1) Art. 112. (2) Art. 116. (3) $\frac{a+b}{2}$, $\sqrt{a\,b}$. (4) Here $a = 7$; $n = 12$; $d = 5$; whence by formula I. $l = 7 + 11 \times 5 = 62$.

GEOMETRICAL PROGRESSION.

116. A series in geometrical progression is one in which each term bears the same ratio to the one preceding, which ratio is called the common ratio.

117. Let a be the first term of a series in geometrical progression, r = the common ratio, n = number of terms, and S = the sum of the series, then

$$S = a + a\,r + a\,r^2 + a\,r^3 + \&c. \ldots + a\,r^{n-1}. \quad \alpha$$

Here multiply both sides of these equal quantities by r; they will remain equal, and we obtain

$$r\,S = a\,r + a\,r^2 + a\,r^3 + \&c. \ldots a\,r^{n-1} + a\,r^n. \quad \beta$$

Subtracting α from β, we have

$$r\,S - S = a\,r^n - a, \text{ or, } (r-1)\,S = a\,(r^n - 1)$$

$$\therefore S = a\,\frac{r^n - 1}{r - 1} \quad \ldots\ldots \quad \text{I.}$$

118. Now when the series decreases, r is a fraction less than unity. In this case it will be more convenient to transform the above formula into

$$S = a\frac{1 - r^n}{1 - r} \;\ldots\ldots\ldots\; \text{II.}$$

Since the last term $l = a\,r^{n-1}$, multiplying by r, we have $r\,l = a\,r^n$, and the expression

$$S = \frac{a\,r^n - a}{r - 1} \text{ becomes } S = \frac{r\,l - a}{r - 1}.$$

119. From this last equation, any three of the letters being given, the fourth may be found. Thus

$$S = \frac{r\,l - a}{r - 1} \;\ldots\ldots\; \text{III.}$$

$$a = r\,l - (r - 1)\,S \;\ldots\ldots\; \text{IV.}$$

$$r = \frac{S - a}{S - l} \;\ldots\ldots\; \text{V.}$$

$$l = \frac{(r - 1)\,S + a}{r} \;\ldots\ldots\; \text{VI.}$$

120. When r is a fraction less than unity, its powers continually diminish, so that the greater the number of terms the smaller will the last term be. Consequently when the number of terms is unlimited, the last term is 0.

Hence, in the case of an infinite series, of which the ratio is a fraction, $S = a\dfrac{1 - 0}{1 - r}$, or $\dfrac{a}{1 - r}$ . . . . VII.

121. But when r is not less than unity no limit can be assigned to the increase of the terms, and the sum of an infinite number of such terms must be infinity.

EXAMPLES.

(1) Find the sum of the geometric series, $3 + 6 + 12 + \ldots\ldots$ &c., to six terms.

Here $a = 3$, $r = 2$, $n = 6$. Substituting these values in I,

$$S = 3\,\frac{2^6 - 1}{2 - 1} = \frac{3 + 64 - 3}{1} = 189. \quad \textit{Ans.}$$

(2) Find the sum of 15 terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}$&c.

Here $a = 1, r = \frac{1}{3}$, $n = 15$. Substituting these values in II,

$$S = \frac{1 - \frac{1}{14348907}}{\frac{2}{3}} = \frac{\frac{14348906}{14348907}}{\frac{2}{3}} = \frac{7174453}{4782969} = 1\,\frac{2391484}{4782969}. \quad \textit{Ans.}$$

(3) Find a geometrical mean between 8 and 128. (*M.* 1848.)

Here $a = 3$, $l = 128$, $n = 3$. Substituting in the equation $l = a r^{n-1}$, $128 = 8 r^2$ $\therefore r^2 = 16$, and $r = 4$.

The mean sought is $8 \times 4 = 32$. *Ans.*

(4) Find the sum of 8, 4, 2, 1, ½, &c., *ad infinitum.*

Here $a=8, r=\frac{1}{2}$. Substituting in VII, $S=8\times\frac{1}{1-\frac{1}{2}}=\frac{8}{\frac{1}{2}}=16$. *Ans.*

(5) Find the sum of $\frac{2}{3}+\frac{2}{3^3}+\frac{2}{3^5}+$ &c., *ad inf.* Here $a=\frac{2}{3}$ $r=\frac{1}{9}$,

$$\therefore S = \frac{2}{3}\times\frac{1}{1-\frac{1}{9}} = \frac{2}{3}\times\frac{1}{\frac{8}{9}} = \frac{2}{3}\times\frac{9}{8} = \frac{3}{4}. \quad \textit{Ans.}$$

(6) Find the sum of $\frac{3}{5^2}+\frac{3}{5^4}+\frac{3}{5^6}+$ &c., *ad inf.* Here $a = \frac{3}{25}$, $r=\frac{1}{25}$, $\therefore S = \frac{3}{25}\times\frac{1}{1-\frac{1}{25}} = \frac{3}{25}\times\frac{25}{24} = \frac{1}{8}$. *Ans.*

(7) Find the sum of the series

$$a - \frac{a}{r} + \frac{a}{r^2} - \ldots\ldots \text{ to } n \text{ terms. } (M.\ 1843.)$$

Here $S = a\,\frac{1-r^n}{1-r}$ becomes

$$S = a\,\frac{1-\left(-\frac{1}{r}\right)^n}{1+\frac{1}{r}}$$

$$= a\,\frac{1\pm\frac{1}{r^n}}{1+\frac{1}{r}} = \frac{a\,r^n \pm a}{r^n + r^{n-1}}$$

Where if n is odd, the positive sign is to be taken, and if n is even, the negative.

(8) Find the limit of the sum of the series

$$\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+ \text{\&c., } \textit{ad infinitum.} \quad (M.\ 1850.)$$

Here $a = \frac{9}{10}$; $r = \frac{1}{10}$; and $S = \frac{a}{1-r}$ becomes

$$S = \frac{\frac{9}{10}}{1-\frac{1}{10}} = \frac{\frac{9}{10}}{\frac{9}{10}} = 1. \quad \textit{Ans.}$$

(9) Find three geometric means between $\frac{2}{3}$ and $\frac{2401}{24}$. (*M.* 1851.)

Let a, $a\,r$, $a\,r^2$, $a\,r^3$, $a\,r^4$, be five terms of a geometric series, then it is plain that the three geometric means between a and $a\,r^4$ are $a\,r$, $a\,r^2$, $a\,r^3$.

But $\frac{a\,r^4}{a} = r^4$, $\therefore \frac{2401}{24} \div \frac{2}{3} = \frac{2401}{16}$ = 4th power of the ratio;

$\therefore$ the ratio $= \frac{7}{2}$, and the three means are $\frac{7}{3}, \frac{49}{6}, \frac{343}{12}$.

(10) The quantities a, x, y, b, are in geometrical progression, find x and y in terms of a and b. (*M.* 1852.)

Since a, x, y, b, are in geometrical progression, we have the quantities $a, a\,r, a\,r^2, a\,r^3$, corresponding to the four given quantities each to each;

$$\therefore x = a\,r;\; y = a\,r^2;\; b = a\,r^3;$$

$$\text{From the last } r = \left(\frac{b}{a}\right)^{\frac{1}{3}}$$

$$\text{Whence } x = a\left(\frac{b}{a}\right)^{\frac{1}{3}} = (a^2\,b)^{\frac{1}{3}},$$

$$y = a\left(\frac{b}{a}\right)^{\frac{2}{3}} = (a^{\frac{1}{2}}\,b)^{\frac{2}{3}}.$$

(11) Find the sums to infinity of the two geometric series $\frac{1}{2} \pm \frac{1}{4} + \frac{1}{8} \pm \frac{1}{16}$, &c.

(When two series are combined as above, the upper signs throughout are to be read first, and then the lower signs throughout.)

$$\text{In both cases } S = \frac{a}{1-r} \;\ldots\ldots\; \text{VII.}$$

$$\text{In the first series } r = \tfrac{1}{2},\; a = \tfrac{1}{2} \;\therefore S = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

$$\text{In the second}\quad r = -\tfrac{1}{2},\; a = \tfrac{1}{2} \;\therefore S = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3}.$$

(12) Investigate the general expression for the sum of n terms of the geometric series $a + a\,r + a\,r^2 + \ldots\ldots$ and (2) find the sum of the series $\frac{1}{5} - \frac{1}{25} + \frac{1}{125}$ — &c., *ad infinitum.* (*M.* 1839.)

(1) Art. 117. (2) Formula VII. $S = \frac{\frac{1}{5}}{1-(-\frac{1}{5})} = \frac{1}{5} \times \frac{5}{6} = \frac{1}{6}$.

(13) To what progression does the series following belong? $1 + \frac{1}{3} + \frac{1}{9}$ + &c. (2) Find its sixth term, and its sum to five terms, and to infinity. (3) Insert two geometric means between ·9 and ·0009. (*M.* 1855.)

Ans.—A series in *geometrical* progression, whose ratio is $\frac{1}{3}$. (2) $\frac{1}{243}$. Its sum by formula II. $= 1 \times \frac{1-(\frac{1}{3})^5}{1-\frac{1}{3}} = \frac{121}{81}$. Its sum to infinity by formula VII. $= \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$. (3) By Art 118,

$l = a\,r^{n-1}$, $\quad r = \sqrt[n-1]{\frac{l}{a}}$; and in this case $n = 4$.

$$\therefore r = \sqrt[3]{\frac{l}{a}} = \sqrt{\frac{\cdot 0009}{\cdot 9}} = \cdot 1.$$

And the four terms are ·9, ·09, ·009, ·0009.

(14) Insert n geometrical means between a and c; and show that their product will be $(a\,c)^{\frac{n}{2}}$. (*M.* 1858.)

Ans.—The ratio is $\left(\frac{c}{a}\right)^{\frac{1}{n+1}}$; and the means are, $a\left(\frac{c}{a}\right)^{\frac{1}{n+1}}$, $a\left(\frac{c}{a}\right)^{\frac{2}{n+1}}$, &c. . . . . $a\left(\frac{c}{a}\right)^{\frac{n-1}{n+1}}$, $a\left(\frac{c}{a}\right)^{\frac{n}{n+1}}$. The product of these is $a^n\left(\frac{c}{a}\right)^{\frac{1+2+3+\,\&c.\,\ldots\,n-1+n}{n+1}}$, or $a^n\left(\frac{c}{a}\right)^{\frac{n+1\cdot n}{2\cdot n+1}}$, which $= a^n\left(\frac{c}{a}\right)^{\frac{n}{2}} = \left(a\,c\right)^{\frac{n}{2}}$.

CHAPTER VI.

SIMPLE EQUATIONS AND QUESTIONS PRODUCING THEM.

SIMPLE EQUATIONS.

122. Two quantities connected by the sign $=$ are called an equation; the two expressions on the right and left of the sign are called sides or members of the equation.

123. Equations are, by some writers, termed *identical* or *formulaic*, *algebraic*, *literal*, and *numerical*.

$(a+b)^2 = a^2 + 2ab + b^2$ is an identical equation, and it is also formulaic, the second number being only an expansion of the first. This equation is true for *any* values of a and b.

$6x = 72 - 3x$ is an algebraic equation, it is only true for *one* value of the unknown quantity x.

An equation is *literal* or *numerical* according as its coefficients are *letters* or *numbers*.

124. When one member of an equation contains *only* the unknown quantity sought, and the other member simply a known number, as $x = 21$, the equation is solved. This form we must always endeavour to attain, however complicated the expression at first proposed may be.

125. Whatever transformations of the original equation may be necessary, they are in all cases founded upon the following obvious axioms:—

"*Two equal quantities will remain equal, whether we add to or subtract from them equal quantities; whether we multiply or divide them by the same number; whether we raise them both to the same powers or extract their roots of the same degree.*"

126. In the equations $x+3=15$, and $x-8=4$, by subtracting 3 from each side of the first equation, and adding 8 to each side of the second, we obtain in each case $x=12$, hence is deduced a rule for facilitating the reduction of equations, that:—

Any quantity may be removed from one side of an equation to the other by changing its sign.

It is also manifest, that any quantity which is found on both sides of an equation may be expunged.

127. Since an equation is not altered by dividing or multiply-

ing each side by the same number, if $2x = 12$ and $\frac{x}{3} = 2$, dividing each side of the first equation by 2, and multiplying the second by 3, then $x = 6$, hence

If the unknown quantity have any coefficient, divide each side by it, and if the unknown quantity have any divisor multiply both sides by it.

128. From what precedes it is manifest that if any part of an equation is fractional, it may be reduced to an equation expressed in integers, by multiplying every term by the denominator of the fraction. An equation containing more than one fraction may be cleared of fractions, by multiplying every term by each denominator in succession, or by their product, or least common multiple.

129. If the unknown quantity be under the radical sign, both sides of the equation must be raised to the same power as denoted by the root; thus—

$$\sqrt[3]{x} = 2\text{; — } \therefore \text{ cubing both sides } x = 8.$$

130. If the radical sign includes a known quantity as well as the unknown quantity, the whole quantity under the radical sign is a surd, and must be brought to one side of the equation; thus if $\sqrt{x+8} - 8 = 3$; first $\sqrt{x+8} = 8 + 3 = 11$, then squaring both sides, $x+8 = 121$ and $x = 113$.

131. *Examples of Simple Equations with one unknown quantity.*

$$(1) \ldots \quad \frac{x}{2} + \frac{x}{3} = 15$$

$$\begin{aligned} 3x + 2x &= 90 \\ 5x &= 90 \\ x &= 18 \textit{ Ans.} \end{aligned}$$

$$(2) \quad \sqrt{15+x} = 3 + \sqrt{x}$$

$$\begin{aligned} 15 + x &= 9 + 6\sqrt{x} + x \\ 6\sqrt{x} &= 6 \\ x &= 1 \textit{ Ans.} \end{aligned}$$

$$(3) \quad \ldots \quad \frac{x+1}{3} - \frac{x+2}{4} = 9 + \frac{x+3}{5}$$

$$\begin{aligned} 20x + 20 - 15x - 30 &= 540 + 12x + 36 \\ 20x - 15x - 12x &= 540 + 36 + 30 - 20 \\ -7x &= 586 \\ x &= -83\tfrac{5}{7} \textit{ Ans.} \end{aligned}$$

$$(4) \ldots\ldots \quad \frac{5}{x} + \frac{6}{x+1} = \frac{11}{x+2}$$

$$\begin{aligned} 5x^2 + 15x + 10 + 6x^2 + 12x &= 11x^2 + 11x \\ 15x + 12x - 11x &= -10 \\ 16x &= -10 \\ x &= -\frac{10}{16} \\ &= -\frac{5}{8} \textit{ Ans.} \end{aligned}$$

(5) . . . . $a\,x + b = a'\,x + b'$. (*M.* 1839.)

$$\therefore (a - a')\,x = b' - b$$

$$x = \frac{b' - b}{a - a'}.$$

(6) . . . $\dfrac{25\,x^2 - 16}{10\,x + 8} = 3\,.\,\dfrac{x^2 - 4}{2\,x - 4}$. (*M.* 1839.)

By actual division, we have—

$$2\,x + \frac{x}{2} - 2 = x + \frac{x}{2} + 3$$

$$x = 5.$$

(7) . . $\dfrac{a\,x^2 + b\,x + c}{a'x^2 + b'\,x + c'} = \dfrac{a\,x + b}{a'x + b'}$. (*M.* 1839.)

By cross-multiplication, we have—

$$a\,a'\,x^3 + (a'\,b + a\,b')\,x^2 + (a'\,c + b\,b')\,x + b'\,c =$$

$$a\,a'\,x^3 + (a'\,b + a\,b')\,x^2 + (a\,c' + b\,b')\,x + b\,c'$$

$$\text{And } (a'\,c - a\,c')\,x = b\,c' - b'\,c$$

$$\therefore x = \frac{b\,c' - b'\,c}{a'\,c - a\,c'}.$$

(8) . . . $\dfrac{x}{a} + \dfrac{x + b}{a + b} + \dfrac{x + c}{a + c} = 3$. (*M.* 1841.)

$$(a^2{+}ab{+}ac{+}bc)x + (a^2{+}ac)x{+}a^2b{+}abc{+}(a^2{+}ab)x{+}a^2c{+}abc$$

$$= 3\,a^3 + 3\,a^2\,b + 3\,a^2\,c + 3\,a\,b\,c;$$

$$\text{or } (3\,a^2 + 2\,a\,b + 2\,a\,c + b\,c)\,x + a^2\,b + a^2\,c + 2\,a\,b\,c$$

$$= 3\,a^3 + 3\,a^2\,b + 3\,a^2\,c + 3\,a\,b\,c;$$

$$\text{or } (3\,a^2 + 2\,a\,b + 2\,a\,c + b\,c)\,x = 3\,a^3 + 2\,a^2\,b + 2\,a^2\,c + a\,b\,c$$

$$x = a.$$

(9). . If $x + 4 : 2\,x - 4 :: 2 : 3$, find x. (*M.* 1857.)

Multiplying extremes and means, we obtain—

$$3\,x + 12 = 4\,x - 8 \;\therefore x = 20.$$

(10) $\dfrac{3\,x - 14}{4} - 5\,x - \dfrac{2\,x - 6}{11} = \dfrac{x}{2} - 72$. (*M.* 1859.)

$$33\,x - 154 - 220\,x - 8\,x + 24 = 22\,x - 3168$$

$$217\,x = 3038;\; x = 14.$$

(11) . $1 + \dfrac{x}{2} - \dfrac{x}{3} = 4 - \dfrac{x + 1}{7} - \dfrac{x - 1}{5}$. (*M.* 1860.)

$$210 + 105\,x - 70\,x = 840 - 30\,x - 30 - 42\,x + 42$$

$$107\,x = 642;\; x = 6.$$

(12) . . $\dfrac{2\,x + 1}{3} - \dfrac{3\,x - 2}{4} = \dfrac{x - 2}{6}$. (*M.* 1861.)

$$8\,x + 4 - 9\,x + 6 = 2\,x - 4;\; x = 4\tfrac{2}{3}.$$

(13) $\frac{x-1}{5}-\frac{x-11}{7}+\frac{3x-(5x-4)}{2}+\frac{278}{35}=0.$ (*M.* 1862.)

$$14x-14-10x+110+105x-175x+140+556=0$$
$$-66x=-792;\ x=12.$$

(14) . $\frac{x+1}{3}-\frac{1}{2}\left(x+3-\frac{3}{2}x\right)=x-2.$ (*M.* 1863.)

$$2x+2-3\left(x+3-\frac{3}{2}x\right)=6x-12$$
$$4x+4-6x-18+9x=12x-24$$
$$-5x=-10;\ x=2.$$

(15) . . . $\frac{2x+7}{x-1}-\frac{x+1}{x+7}=1.$ (*M.* 1863.)

$$2x^2+21x+49-x^2+1=x^2+6x-7.$$
$$15x=-57.$$
$$x=-3\tfrac{4}{5}.$$

Simple Equations, with two or more unknown quantities.

132. Determinate equations have the same number of distinct equations as there are unknown quantities required. Indeterminate equations are such as have a less number of equations than unknown quantities.

133. Let (I) $5x+6y=76$, and (II) $4x-3y=14$. An infinite number of values may be given to x and y in both these equations, taken separately, thus—

$$\begin{array}{ll}
\text{I. If } x=1,\ y=11\tfrac{5}{6} & \text{II. If } x=1,\ y=-3\tfrac{1}{3} \\
\quad =2,\ y=11 & \quad =2,\ y=-2 \\
\quad =3,\ y=10\tfrac{1}{6} & \quad =3,\ y=-\tfrac{2}{3} \\
\quad \&c. & \quad \&c. \\
\text{Or if } x=-1,\ y=13\tfrac{1}{2} & \text{Or if } x=-1,\ y=-6 \\
\quad =-2,\ y=14\tfrac{1}{3} & \quad =-2,\ y=-7\tfrac{1}{3} \\
\quad =-3,\ y=15\tfrac{1}{6} & \quad =-3,\ y=-8\tfrac{2}{3} \\
\quad \&c. & \quad \&c.
\end{array}$$

134. Continuing, as indicated above, to assume positive and negative values for x in both equations, we find

I. If $x=8,\ y=6$, and also II. If $x=8,\ y=6$.

In this case, both equations are solved by the same values of x and y. It is evident also, on proceeding further as above, that these are the *only* values of x and y which solve *both* equations.

135. Equations thus solved, by the same values of x and y, are called *simultaneous equations.*

136. There are three methods of solving equations with two unknown quantities. Equations may be solved by any of the three methods; the second is most generally applicable.

FIRST METHOD.

Find a value of x or y in either equation, (whichever is most convenient,) and substitute it in the other, the resulting equation will have only one unknown quantity.

$$\text{Given } 3x + 11y = 84.$$
$$7x - 19y = 62.$$

From first equation $x = \frac{84 - 11y}{3}$, substituting this value of x in second, $7 + \frac{84 - 11y}{3} - 19y = 62$; clearing this equation of fractions, $588 - 77y - 57y = 186$

$$\text{or } 134y = 402$$
$$\left.\begin{aligned} y &= 3 \\ x &= 17 \end{aligned}\right\} \textit{Ans.}$$

The value of x is obtained by substituting the value of y in either of the original equations.

SECOND METHOD.

Eliminate x or y from both equations by bringing the coefficients of either to a common coefficient, which can be done by finding their least common multiple; or as it is more generally done, by multiplying each equation by the coefficient in the other equation belonging to the quantity to be eliminated. Then adding the two equations, as is done in Ex. 2, or subtracting one from the other, as in Ex. 1, a new equation arises, as before, involving only one unknown quantity.

Ex. 1.

$$\begin{array}{r} 4x + 9y = 51 \\ 8x - 13y = 9 \\ \hline 8x + 18y = 102 \\ 8x - 13y = 9 \\ \hline 31y = 93 \\ y = 3 \end{array}$$

From 2nd equa. $8x - 13y = 9$

$$\text{but } y = 3.$$
$$\therefore 8x - 39 = 9$$
$$8x = 48$$
$$x = 6$$
$$\left.\begin{aligned} x &= 6 \\ y &= 3 \end{aligned}\right\} \textit{Ans.}$$

Ex. 2.

$$\begin{array}{r} 5x + 6y = 137 \\ 13x - 4y = 23 \\ \hline 10x + 12y = 274 \\ 39x - 12y = 69 \\ \hline 49x \qquad = 343 \\ x = 7 \end{array}$$

From 1st equa. $5x + 6y = 137$

$$\text{but } x = 7.$$
$$\therefore 35 + 6y = 137$$
$$6y = 102$$
$$y = 17$$
$$\left.\begin{aligned} x &= 7 \\ y &= 17 \end{aligned}\right\} \textit{Ans.}$$

Third Method.

Find the value of x or y in each equation, and equate these values of x or y, by which an equation is obtained containing only one unknown quantity.

$$\text{Given} \quad \begin{aligned} 5x + 6y &= 76 \quad (1) \\ 4x - 3y &= 14 \quad (2) \end{aligned}$$

$$\text{From (1) } x = \frac{76 - 6y}{5} \qquad \text{From (2) } x = \frac{14 + 3y}{4}$$

$$\therefore \frac{76 - 6y}{5} = \frac{14 + 3y}{4}$$

$$\begin{aligned} 304 - 24y &= 70 + 15y \\ 39y &= 234 \\ \therefore y &= 6 \end{aligned}$$

Substituting this value of y in the second equation, we have—

$$\begin{aligned} 4x - 18 &= 14 \\ 4x &= 32 \\ \therefore x &= 8 \end{aligned}$$

This method may frequently be applied still more advantageously by obtaining the same coefficients to x or y, instead of the initial values of x or y.

$$\text{Given} \quad \begin{aligned} 4x + 7y &= 62 \quad (1) \\ 3y - 2x &= 8 \quad (2) \end{aligned}$$

Multiplying (2) by -2, we have—

$$4x = 6y - 16, \text{ and from (1) } 4x = 62 - 7y.$$

$$\begin{aligned} \therefore 6y - 16 &= 62 - 7y \\ 13y &= 78 \\ y &= 6 \end{aligned} \qquad \begin{aligned} \text{From (2) } 3y - 2x &= 8 \\ \therefore 18 - 2x &= 8 \\ x &= 5 \end{aligned}$$

$$\left.\begin{aligned} x &= 5 \\ y &= 6 \end{aligned}\right\} \textit{Ans.}$$

Examples.

⁂ *Throughout the following solutions, steps and explanations given above, are intentionally omitted, to be supplied by the student.*

(1) (*M.* 1838.) $\left\{\begin{aligned} \frac{3}{x} + \frac{3}{y} &= 6 \\ x + y &= 2 \end{aligned}\right.$ From the 1st, $x + y = 2xy$. By subtracting the 2nd, $2xy = 2$.

$$\therefore xy = 1.$$

$$\begin{aligned} \text{Again } x^2 + 2xy + y^2 &= 4 \\ 4xy &= 4 \\ \hline x^2 - 2xy + y^2 &= 0 \end{aligned}$$

$$x - y = 0, \therefore x = 1, \text{ and } y = 1.$$

(2) (*M.* 1839.)

$$\left\{\begin{array}{l} \dfrac{11x-5y}{22}=\dfrac{3x+y}{32}, \quad \text{or } 176x-80y=33x+11y \\ \qquad 8x-5y=1 \quad ,,\ 11x-7y=0 \end{array}\right.$$

$$\begin{array}{rl} 56x-35y &= 7 \\ \underline{55x-35y} &\underline{= 0} \\ x &= 7 \quad \therefore y=11. \end{array}$$

(3) (*M.* 1854.)

$$\left\{\begin{array}{ll} \dfrac{x}{2}+\dfrac{y}{3}=29 & \text{or } 6x+4y=348 \\ \dfrac{x}{3}-\dfrac{y}{2}=2 & ,,\ \underline{6x-9y=36} \end{array}\right.$$

$$13y=312$$

$$y=24$$

$$\left.\begin{array}{r} \therefore x=42 \\ y=24 \end{array}\right\} \textit{Ans.}$$

(4) (*M.* 1855.)

$$\left\{\begin{array}{l} 3x-2y=11, \text{ or } 12x-8y=44 \\ 4x+7y=63 \quad ,,\ \underline{12x+21y=189} \end{array}\right.$$

$$29y=145$$

$$y=5$$

$$\left.\begin{array}{r} \therefore x=7 \\ y=5 \end{array}\right\} \textit{Ans.}$$

(5) (*M.* 1855.)

$$\left\{\begin{array}{ll} 2x-\dfrac{3x-y}{7}=15+\dfrac{x}{2} & \text{or } 28x-6x+2y=210+7x \\ 3y-\dfrac{5x-4}{8}=\dfrac{x+y}{3}+29 & ,,\ 72y-15x+12=8x+8y+696. \end{array}\right.$$

$$\text{Again}\quad 15x+2y=210, \text{ or } 480x+64y=6720$$

$$-23x+64y=684 \quad ,,\ \underline{-23x+64y=684}$$

$$503x=6036, x=12.$$

$$\left.\begin{array}{r} \therefore x=12 \\ y=15 \end{array}\right\} \textit{Ans.}$$

(6) (*M.* 1856.)

$$\left\{\begin{array}{l} 3x-7y=7, \text{ or } 15x-35y=35 \\ 11x+5y=87 \quad ,,\ \underline{77x+35y=609} \end{array}\right.$$

$$92x=644 \therefore x=7, \text{and } y=2.$$

(7) (*M.* 1856.)

$$\left\{\begin{array}{ll} \dfrac{x}{9}+\dfrac{y}{8}\quad 43, & \text{or } 8x+9y=43\times 72 \quad (1) \\ \dfrac{x}{8}+\dfrac{y}{9}=42 & ,,\ \underline{9x+8y=42\times 72} \quad (2) \end{array}\right.$$

$$17x+17y=85\times 72$$

$$x+y=5\times 72=360.$$

$$\text{Subtract (2) from (1)}\ -x+y=72$$

$$\underline{x+y=360}$$

$$2y=432 \text{ and } y=216 \therefore x=144.$$

(8) (*M.* 1857.)

$$\left\{\begin{array}{ll} \dfrac{x}{3}+\dfrac{y}{5}=8, & \text{or } \dfrac{x}{3}+\dfrac{2y}{10}=8 \\ \dfrac{y}{9}-\dfrac{y}{10}=1 & ,,\ \dfrac{x}{3}-\dfrac{3y}{10}=3 \end{array}\right.$$

$$\frac{5y}{10}=5$$

$$\therefore y=10, \text{ and } x=18.$$

(9) (*M.* 1858.) $$\begin{cases} 2x+3y=61 \ (1) \\ 5x-4y=26 \ (2) \end{cases} \quad \text{Now } (1)\times 4+(2)\times 3 \text{ gives } 23x=322 \quad \therefore x=14 \text{ and } y=11.$$

$$(3x+8)=4y-4=2(x+y-1) \quad (M.\ 1859.)$$

$$\begin{array}{rll} 3x+8 & =4y-4, & \text{or } 4y-3x=12 \\ 2x+2-2 & =4y-4 & ,, \ \underline{4y-4x=\ 4} \\ & & x=\ 8 \text{ and } y=9. \end{array}$$

(10) (*M.* 1860.) $$\begin{cases} \dfrac{3x+2y-3}{4x-5y+16}=\dfrac{9}{4}=24x-53y=-156 & (1) \\ 3x=5y \text{ or } 3x-5y=0 & (2) \end{cases}$$

$$(2)\times 8-(1) \text{ gives } 18y=156 \ \therefore y=12 \text{ and } x=20.$$

(*M.* 1860.) $$\begin{cases} \dfrac{x+1}{y-1}+\dfrac{y}{x-4}=5 \ (1) & \text{Now } (1)\times 3+(2)=4.\dfrac{x+1}{y-1}= \\ & 16 \text{ or } \dfrac{x+1}{y-1}=4 \\ \dfrac{x+1}{y-1}-3\ \dfrac{y}{x-4}=1 \ (2) & \therefore x-4y=-5. \quad (3) \end{cases}$$

Equations with Three Unknown Quantities.

137. If there be three unknown quantities required, it is necessary to have three separate equations; the solution is then obtained by an extension of the previous rules, as follows:—

Either first, eliminate x from any two equations, and then eliminate x from the third and either of the former. From these processes two new equations are obtained, containing only two unknown quantities, the values of which are found as before. The value of the third unknown quantity is found by substituting, in any of the three given equations, the values of the two unknown quantities already found.

Or, secondly, find a value of x from each equation, any two of which may then be reduced to one, containing only two of the unknown quantities, and then let the third and either of the former be reduced to one containing the same unknown quantities. In this way two new equations are obtained, containing only two unknown quantities, and their values are found as has been already described.

Equations containing four or more unknown quantities, require for their solution four or more independent equations. The previous rules are applicable and sufficient for all ordinary cases.

EXAMPLES.

(1) (*M.* 1848.) $$\begin{cases} 2x+y+z=13 & (1) \\ x+2y+3z=16 & (2) \\ 3x-2y+4z=14 & (3) \end{cases}$$

$$\begin{array}{ll}
(2)\ 2x + 4y + 6z = 32 & (2)\ 6x + 12y + 18z = 96 \\
(1)\ \underline{2x + \ \ y + \ \ z = 13} & (1)\ \underline{6x - 4y + 8z = 28} \\
\quad 3y + 5z = 19\ (4) & \quad 16y + 10z = 68\ (5)
\end{array}$$

$$\begin{array}{rl}
(5) & 16y + 10z = 68 \\
(4) & \underline{\ 6y + 10z = 38} \\
 & 10y \qquad\ \ = 30 \quad \therefore y = 3.
\end{array}$$

By substituting the value of y in (4) we get $z = 2$; and the values of y and z in (2) $x = 4$.

$$\left.\begin{array}{r} \therefore x = 4 \\ y = 3 \\ z = 2 \end{array}\right\} \textit{Ans.}$$

(2)

$$\left\{\begin{array}{ll}
\frac{1}{x} + \frac{1}{y} = 1 & (1) \\
\frac{1}{x} + \frac{1}{z} = 2 & (2) \\
\frac{1}{y} + \frac{1}{z} = \frac{3}{2} & (3)
\end{array}\right.
\qquad
\begin{array}{l}
\text{Now } (2) - (1),\ \frac{1}{z} - \frac{1}{y} = 1 \quad (4) \\
\text{,, } (4) + (3),\ \frac{2}{z} = \frac{5}{2} \\
\text{,, } (3) - (4),\ \frac{2}{y} = \frac{1}{2} \quad \therefore y = 4.
\end{array}$$

(*M.* 1839.)

$$\left.\begin{array}{r} \therefore x = 1\frac{1}{3} \\ y = 4 \\ z = \frac{4}{5} \end{array}\right\} \textit{Ans.}$$

(3)

$$\left\{\begin{array}{ll}
7x - 3y = 1 & (1) \\
4z - 7y = 1 & (2) \\
11z - 7u = 1 & (3) \\
19x - 3u = 1 & (4)
\end{array}\right.
\qquad
\begin{array}{rl}
\text{From } (4) & 532x - 84u = 28 \\
(1) & \underline{532x - 228y = 76} \\
 & 228y - 84u = -48 \\
(3) & \underline{132z - 84u = \ \ 12} \\
 & 228y - 132z = -60 \\
(2) & \underline{-231y + 132z = \ \ 33} \\
 & -3y \qquad\quad = -27 \\
 & \qquad y = 9.
\end{array}$$

(*M.* 1839.)

Substituting the value of y in (1) $x = 4$; the value of x in (4) $u = 25$; and similarly from (2) $z = 16$.

$$\left.\begin{array}{r} \therefore x = 4 \\ y = 9 \\ u = 25 \\ z = 16 \end{array}\right\} \textit{Ans.}$$

(4)

$$\left\{\begin{array}{ll}
\frac{x}{a} + \frac{y}{b} = 1, \text{ or } bx + ay = ab & (1) \\
\frac{x}{a'} + \frac{z}{c} = 1, \text{ or } cx + a'z = a'c & (2) \\
\frac{y}{b'} + \frac{z}{c'} = 1, \text{ or } c'y + b'z = b'c' & (3)
\end{array}\right.$$

(*M.* 1839.)

$$\begin{array}{ll}
(1) \times c & b\,c\,x + a\,c\,y = a\,b\,c \\
(2) \times b & b\,c\,x + a'\,b\,z = a'\,b\,c \\
\hline
 & a\,c\,y - a'\,b\,z = a\,b\,c - a'\,b\,c \qquad (4) \\
(3) \times ac & a\,c\,c'\,y + a\,b'\,c\,z = a\,b'\,c\,c' \\
(4) \times c' & a\,c\,c'\,y - a'\,b\,c'\,z = a\,b\,c\,c' - a'\,b\,c\,c' \\
\hline
 & (a\,b'\,c + a'\,b\,c')\,z = a\,b'\,c\,c' - a\,b\,c\,c' + a'\,b\,c\,c'
\end{array}$$

$$z = \frac{a\,b'\,c\,c' - a\,b\,c\,c' + a'\,b\,c\,c'}{a\,b'\,c + a'\,b\,c'}.$$

$$\text{From (3) } \frac{y}{b'} + \frac{a'\,b\,c - a\,b\,c + a'\,b\,c}{a\,b'\,c + a'\,b\,c'} = 1.$$

$$y = \frac{a\,b\,b'\,c - a'\,b'\,b\,c + a'\,b'\,b\,c'}{a\,b'\,c + a'\,b\,c'}.$$

$$\text{From (1) } \frac{x}{a} + \frac{a\,b'\,c - a'\,b'\,c + a'\,b'\,c'}{a\,b'\,c + a'\,b\,c'} = 1.$$

$$x = a - \frac{a\,b'\,c - a\,a'\,b'\,c + a\,a'\,b'\,c'}{a\,b'\,c + a'\,b\,c'}.$$

$$x = \frac{a\,a'\,b\,c' + a\,a'\,b'\,c - a\,a'\,b'\,c'}{a\,b'\,c + a'\,b\,c'}.$$

PROBLEMS PRODUCING SIMPLE EQUATIONS, &c.

138. The solution of problems depends upon two distinct operations, *viz.*, the formation of one, or more than one, equation from the problem, and the solution of the equations thus formed. The latter operation has been already fully explained and illustrated for Simple Equations, and will again be treated of for quadratics. We shall, therefore, in the following solutions, chiefly confine our attention to the former operation.

Instead of repeating such phrases as "according to the terms of the question," or, "by the conditions laid down in the problem," this abbreviation (*p. q.*) will be used.

EXAMPLES.

(1) Wishing to buy a certain number of railway shares, I found that if I bought the shares in the railway (A) which were at £40 a share, I should invest all my money; but if I bought the same number of shares in a railway (B) which were at £45 a share, I should not have money enough by £240. How much money had I to invest? (*M.* 1859.)

Let x = number of shares, then

$$\left.\begin{array}{r}(p.\,q.)\ 40\,x\\ \text{and } 45\,x-240\end{array}\right\}=\text{all my money};$$

$$\therefore 45\,x - 240 = 40\,x$$

$$5\,x = 240$$

$$x = 48 \text{ and } 40\,x = \pounds 1{,}920. \quad \textit{Ans.}$$

(2) On Monday, June 9th, the turnstiles recorded the entrance into the Exhibition of 58,682 persons. The money taken (in shillings) consisted of a number of pounds and 7 shillings over, and it was observed that the number of pounds was less by 295 than the number of persons who entered with season tickets. Find the number of persons who entered, respectively, by payment and by season tickets. (*M.* 1862.)

Let x = number of season tickets, then

$$(p.\,q.)\ x - 295 = \text{number of pounds taken},$$

$$\left.\begin{array}{r}58{,}682 - x\\ 20\,(x - 295) + 7\end{array}\right\}=\text{number of shillings taken};$$

$$\therefore 20\,x - 5{,}900 + 7 = 58{,}682 - x$$

$$21\,x = 64{,}575$$

$$x = 3{,}075.$$

$$\textit{Ans.}\ \left\{\begin{array}{ll}3{,}075 & \text{entered by season tickets,}\\ 55{,}607 & \quad ,, \quad \text{payment.}\end{array}\right.$$

(3) It is required to divide 36 into three such parts, that one-half of the first, one-third of the second, and one-fourth of the third may be equal to each other. (*M.* 1863.)

Let x be any of the three fractions, then

$$(p.\,q.) \text{ the parts are } (2\,x + 3\,x + 4\,x) = 9\,x = 36;$$

$$\therefore x = 4 \text{ and the numbers are}$$

8, 12, 16.

(4) What is the value of x, when £x and 18 shillings are twice the amount of £18 and x shillings precisely? (*M.* 1842.)

$$\text{First } (p.\,q.)\ \pounds x + 18s. = 2\,(\pounds 18 + x\,s.)$$

$$\text{or } 20\,x + 18 = 720 + 2\,x$$

$$18\,x = 702 \text{ and } x = 39.$$

(5) Two railroad trains start at the same time, one from London at the rate of 25 miles an hour, the other from Bristol at 30 miles an hour: the distance from London to Bristol is 120 miles; at what distance from Bristol will the trains meet? (*M.* 1843.)

The *two* trains go 55 miles in 1 hour, and if x = time of meeting, we have $(p.\,q.)$

$$x = \frac{120}{55} \text{ hrs.} = 2 \text{ hrs. } 10\tfrac{10}{11} \text{ min.}$$

$$\begin{array}{lr}\text{The train from London has completed} & 54\tfrac{36}{66}\\ \quad ,, \quad ,, \quad \text{Bristol} \quad ,, \quad ,, & 65\tfrac{30}{66}\\ \hline & 120 \text{ miles.}\end{array}$$

(6) At an election three candidates stood, and 1,000 votes were polled. The second candidate on the poll had twenty votes more than the third; but if the first had only polled the same number of votes as the second, and the same number of votes had been polled, the third candidate would have been at the head of the poll by a majority of 10. How many votes were polled by each? (*M.* 1844.)

$$\left.\begin{array}{rl} \text{Let } x = & \text{No. of votes polled for the 1st. c.} \\ y = & \quad ,, \qquad ,, \qquad \text{2nd.} ,, \\ y - 20 = & \quad ,, \qquad ,, \qquad \text{3rd.} ,, \end{array}\right\} \text{ then we have } (p.\ q.)$$

$$\begin{array}{lll} x + 2y - 20 & = 1000, \text{ or } x + 2y & = 1020 \\ \text{and } 3y + 10 & = 1000 \quad ,, \qquad y & = \ 330 \\ & \qquad\qquad\quad x & = \ 360 \end{array}$$

$$\textit{Ans.} \left\{\begin{array}{l} 360 \text{ voted for the 1st.} \\ 330 \quad ,, \quad ,, \quad \text{2nd.} \\ 310 \quad ,, \quad ,, \quad \text{3rd.} \end{array}\right.$$

(7) A railroad 20 miles long is on a gentle slope; a train can travel along it and return in 40 minutes, and it is found that the train performs 3 miles down the slope in the same time that it takes to perform 2 miles up it. Required the time it takes to go and to return, and the rate of travelling each way. (*M.* 1846.)

$$\begin{array}{l} \text{Let } 2x = \text{No. of min. going down,} \\ \therefore 3x = \quad ,, \quad ,, \quad ,, \ \text{up,} \\ (p.\ q.)\ 5x = 40; \ \therefore x = 8, \text{ and} \\ \therefore 16 \text{ min.} = \text{time in going down,} \\ 24 \quad ,, \quad ,, \quad \text{returning.} \end{array}$$

$$\begin{array}{lllll} \text{min.} & \text{min.} & \text{miles.} & \text{miles.} & \\ 16 : & 60 :: & 20 : & 75 & \text{rate in going down,} \\ 24 : & 60 :: & 20 : & 50 & \quad ,, \quad ,, \quad \text{up.} \end{array}$$

(8) A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head; after being at sea 20 days, she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to $\frac{5}{7}$ths of a pound. Find the original number of the crew. (*M.* 1858.)

Let x = number of the crew, then the ship sailed with $60\,x$ lbs. of biscuit, and at the end of 20 days $(60 - 20)\,x = 40\,x$ lbs. remained to serve $(x - 5)$ men for 64 days, and finally

$$\begin{array}{ll} \dfrac{40\,x}{64\,(x - 5)} & = \dfrac{5}{7}, \text{ or} \\ 280\,x & = 320\,x - 1600; \\ \therefore x & = 40. \quad \textit{Ans.} \end{array}$$

(9) A railroad train travels at the rate of 24 miles an hour; two hours after it has started, an express engine, travelling at the rate of 40 miles an hour, is sent to overtake it. After what time, and what number of miles will the express come up with the train? (*M.* 1847.)

Let $x =$ number of hours first train travels,
$x - 2 =$,, ,, express ,,
(*p. q.*) $24\,x = 40\,x - 80$ or $16\,x = 80$;
$\therefore x = 5$ and $x - 2 = 3$ hours. *Ans.*

(10) Ten years since, the ages of A and B together exceeded four times C's age by six years, but in ten years hence that sum will only exceed three times C's age by three years; when C was born, A was exactly three times the age of B,—what are their present ages? (*M.* 1841.)

First let $x = C$'s age, and $y = B$'s age when C was born.

$$(p.q.)\ \text{10 yrs. since ages of} \begin{cases} C = x - 10 \\ (A + B) = 4\,x - 40 + 6 = 4\,x - 34. \end{cases}$$

,, ,, hence ,, $(A + B) = 4\,x - 34 + 40 = \underline{4\,x + 6}.$
,, ,, ,, age of $C = x + 10; \therefore 3\,C = \overline{3\,x + 30}.$

Again (*p. q.*) $4\,x + 6 = \overline{3\,x + 30} + 3$;
$\therefore x = 27$, and C's age 10 years ago was 17;
$\therefore (A + B)$'s ages ,, ,, 74.

Also if B's age when C was born $= y$, then
(*p. q.*) A's ,, ,, ,, $= 3\,y$, and
,, $(A + B)$'s ages ,, ,, $= 74 - (17 \times 2) = 40$; but
,, $(A + B)$'s ,, ,, ,, $= 3\,y + y = 4\,y$;
$\therefore 4\,y = 40$ and $y = 10$.

$$\text{Finally } (p.q.)\ \left.\begin{array}{l} C\text{'s age} = 27 \\ B\text{'s ,,} = y + 27 = 37 \\ A\text{'s ,,} = 3\,y + 27 = 57 \end{array}\right\}\ \textit{Ans.}$$

(11) A courier undertakes to perform a journey on foot of 60 miles within 12 hours; he travels $6\frac{1}{2}$ miles the first hour, but afterwards in every successive hour he travels $\frac{1}{4}$ of a mile less than in the preceding hour. Will he perform his undertaking? (*M.* 1860.)

The spaces he travels in each hour form a series in arithmetical progression, of which the ratio is $-\frac{1}{4}$.

$$6\tfrac{1}{2} + 6\tfrac{1}{4} + 6 + \&c. \ldots \text{to 12 terms} = \frac{(6\frac{1}{2} + 3\frac{3}{4})12}{2} = 10\tfrac{1}{4} \times 6 = 61\tfrac{1}{2}\ \text{miles.}$$

Ans. Yes: with the time of going $1\frac{1}{2}$ miles to spare.

CHAPTER VII.

ALGEBRAIC PROPORTION AND VARIATION, SIMPLE INTEREST, DISCOUNT, AND STOCKS.

139. All the important propositions of the fifth book of Euclid are proved, for any commensurable quantities, in this section, and references to them given in the margin.

The remarks on arithmetical proportion apply to any numbers or quantities whatever, and, consequently, letters may be substituted for the numbers there assumed.

To make the statement of the following propositions more easily understood, numbers are first employed. In each succeeding proposition, the same proportion of numbers and letters is assumed, as in the first proposition.

140. If $20 : 8 :: 15 : 6$ or if $a : b :: c : d$, then

I.—*Alternando*. [*Eu.* V., 16.

$$20 : 15 :: 8 : 6 \qquad a : c :: b : d.$$

$$\text{For } \frac{a}{b} = \frac{c}{d} \therefore \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}; \text{ or, } \frac{a}{c} = \frac{b}{d};$$

i.e. $a : c :: b : d$.

This change in the proportion, by which antecedent is compared with antecedent, and consequent with consequent, is termed ***alternando vel permutando***, by alternation or permutation, or more simply, alternate proportion.

II.—*Invertendo*. [*Eu.* V., B.

$$8 : 20 :: 6 : 15 \qquad b : a :: d : c.$$

141. Since the quantities themselves are proportionals, their *reciprocals* are so too.

Note.—Unity divided by any quantity gives the reciprocal of the quantity, *e.g.*, $1 \div \frac{4}{7} = \frac{7}{4}$, the reciprocal of $\frac{4}{7}$. Any quantity multiplied by its reciprocal gives unity. To divide by a quantity or multiply by its reciprocal produces the same result.

$$\text{Hence if } \frac{a}{b} = \frac{c}{d} \text{ then } \frac{b}{a} = \frac{d}{c},$$

i.e. $b : a :: d : c$.

In this proportion the antecedent becomes the consequent, and the consequent is made the antecedent, or the ratios are *inverted;* hence this change is called *invertendo,* by inversion, or simply, inverted proportion.

III.—*Componendo.* [*Eu.* V., 18.

142. $28 : 20 :: 21 : 15$ or, $a + b : a :: c + d : c$
Or, $28 : 8 :: 21 : 6$ or, $a + b : b :: c + d : d$

This is compounded ratio, in which the sum of the antecedent and consequent is compared with either the antecedent or consequent.

IV.—*Dividendo.* [*Eu.* V., 17.

143. $12 : 20 :: 9 : 15$ or, $a - b : a :: c - d : c$
Or, $12 : 8 :: 9 : 6$ or, $a - b : b :: c - d : d$

Divided ratio, in which the difference of antecedent and consequent is compared with antecedent or consequent.

V.—*Convertendo.* [*Eu.* V., E.

144. $20 : 28 :: 15 : 21$ or, $a : a \pm b :: c : c \pm d$
$20 : 12 :: 15 : 9$

Converse proportion, in which the antecedents are compared with the sums or differences of the antecedents and consequents.

Proof of III., IV., *and* V.

For $\frac{a}{b} = \frac{c}{d}$ $\therefore \frac{a}{b} \pm 1 = \frac{c}{d} \pm 1$, i. e. $\frac{a \pm b}{b} = \frac{c \pm d}{d}$;

Or, $a \pm b : b :: c \pm d : d$.

Again, it has been shown *invertendo,* $\frac{b}{a} = \frac{d}{c}$, and since $\frac{a \pm b}{b} = \frac{c \pm d}{d}$ $\therefore$ by multiplication $\frac{(a \pm b)\, b}{a\, b} = \frac{(c \pm d)\, d}{c\, d}$;

$\therefore a \pm b : a :: c \pm d : c$.

Lastly, *invertendo,* $a : a \pm b :: c : c \pm d$.

145. *Cor.* Since $\frac{a + b}{b} = \frac{c + d}{d}$; and $\frac{a - b}{b} = \frac{c - d}{d}$, dividing the first by the second, $\frac{a + b}{a - b} = \frac{c + d}{c - d}$, i. e., $a + b : a - b :: c + d : c - d$.

VI. [*Eu.* V. Def. 5.

146. . . . . . . $m\,a : m\,b :: m\,c : m\,d$
$m\,a : n\,b :: m\,c : n\,d$

$\because \frac{a}{b} = \frac{c}{d}$, multiplying both fractions successively by $\frac{m}{m}$ and $\frac{m}{n}$, we obtain, $\frac{m\,a}{m\,b} = \frac{m\,c}{m\,d}$, and $\frac{m\,a}{n\,b} = \frac{m\,c}{n\,d}$;

$$\text{Or, } m\,a : m\,b :: m\,c : m\,d$$
$$\text{And, } m\,a : n\,b :: m\,c : n\,d$$

If four quantities be proportional, and any equi-multiples or equi-submultiples of the antecedents be taken, and also of the consequents, these products are likewise proportional.

$\therefore$ if $m\,a$ is $>, =,$ or $< n\,b$; $m\,c$ is $>, =,$ or $< n\,d$, the algebraic form of Euclid's 5th definition, Book V.

VII.

147. . . . $a \pm^{n} : b \pm^{n} :: c \pm^{n} : d \pm^{n}$

$$\because \frac{a}{b} = \frac{c}{d} \quad \therefore \frac{a \pm^{n}}{b \pm^{n}} = \frac{c \pm^{n}}{d \pm^{n}}, \text{ or,}$$

$$a \pm^{n} : b \pm^{n} :: c \pm^{n} : d \pm^{n}.$$

If four quantities be proportional, the like powers or roots of those quantities are proportional.

VIII. [*Eu.* V., 25.

148. If four quantities be proportional, the sum of the greatest and least is greater than the sum of the other two. a being the greatest of the four terms in a proportion, d will be the least, and $(a + d) > (b + c)$

$$\because a > b \quad \therefore c > d$$
$$\because a > c \quad \therefore b > d$$
$$\therefore d \text{ is the least of the four.}$$

$$\text{Again, } \frac{a}{b} = \frac{c}{d}$$
$$\therefore (140.)\ \frac{a}{c} = \frac{b}{d}$$
$$\therefore (143.)\ \frac{a-c}{c} = \frac{b-d}{d}$$
$$\therefore (140.)\ \frac{a-c}{b-d} = \frac{c}{d}$$
$$\text{But } c > d \quad \therefore a - c > b - d$$
$$\therefore a - c + d > b$$
$$\therefore a + d > b + c.$$

149. *Scholia.* I. When the ratio of two numbers or quantities, $\frac{a}{b}$, cannot be expressed in numbers they are said to be incommensurable.

150. II. The existence of incommensurable quantities is de

monstrated by Euclid in book X., prop. 117. His method of reasoning may be represented as follows:

Draw a square bisected by its diagonal, and, if possible, let the diagonal and a side have the common measure m, and let the diagonal $= a\,m$ and the side $= b\,m$, and lastly, let a and b have no common measure but unity, or if they have, divide both by it, and their relative value will remain the same; then, by the 47th proposition of the first book of Euclid,—

$$a^2\,m^2 = 2\,b^2\,m^2 \quad \therefore a^2 = 2\,b^2.$$

Now it is evident that $2\,b^2$ must be an even number, therefore a^2 is an even number; and since the product of two odd numbers is always an odd number, and the product of two even numbers is always an even number, therefore a is an even number, and then b must be an odd number, or a and b would be divisible by 2, which is contrary to the hypothesis.

Next let $a = 2\,k$, and k will be a whole number because a is even,

$$\text{then } a^2 = 4\,k^2 = 2\,b^2;$$

$$\therefore 2\,k^2 = b^2.$$

Again, $2\,k^2$ must be an even number, and therefore b^2, and consequently b is an even number; but b was proved to be an odd number. Hence the same number is both odd and even, which is impossible. Therefore m is not the common measure of a and b, *i.e.*, the diagonal and side of a square have no common measure except unity.

151. III. Although the ratio of two incommensurable quantities, a and b, cannot be expressed in numbers, a fraction can always be found as near to that ratio as we please.

For if $b = n\,x$, and a does not contain the x exactly,

$$\text{suppose } a > m\,x, \text{ and } < \overline{m + 1}.x,$$

$$\text{then } \frac{a}{b} > \frac{m}{n}, < \frac{m+1}{n}; \text{ and } \frac{m+1}{n} - \frac{m}{n} = \frac{1}{n};$$

$\therefore$ the ratio of $a : b$ does not differ from that of $m : n$ by so much as the ratio $1 : n$; which by making n large enough may be made as small as we please.

EXAMPLES.

(1) Explain when four quantities are said to be in proportion. (2) Show that if $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$, and (3) $a^n : b^n :: c^n : d^n$. (*M.* 1838 and 1839.)

(1) Art. 101. (2) Art. 145. (3) Art. 147.

(2) Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$. (2) And also show that if a be less than b, the fraction $\frac{a}{b}$ will be less than $\frac{a+x}{b+x}$ when x is any positive quantity. (*M.* 1841.)

(1) Art. 145. (2) $\frac{a+x}{b+x}-\frac{a}{b}=\frac{a\,b+b\,x-a\,b-a\,x}{b^2+b\,x}=\frac{(b-a)\,x}{b^2+b\,x}$, and this is a positive quantity since $b>a$;

$\therefore \frac{a+x}{b+x}>\frac{a}{b}$ by $\frac{(b-a)\,x}{b^2+b\,x}$, when x is any positive quantity.

(3) Show that if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$, then will $\frac{a}{b}=\frac{a+c+e+\ldots}{b+d+f+\ldots}$ (*M.* 1842 and 1849.)

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots m$, then will $a=b\,m$, $c=d\,m$, $e=f\,m\ldots$

$$\text{and } a+c+e+\ldots\ldots=(b+d+f+\ldots)\,m$$

$$\therefore \frac{a+c+e+\ldots}{b+d+f+\ldots}=m=\frac{a}{b}.$$

(4) Define proportion. (2) If four quantities be proportional, the product of the extremes is equal to the product of the means. (3) What well known rule in arithmetic is founded on this property? (4) If the four magnitudes be four straight lines, how is this result interpreted? (5) If $a:b::c:d$ prove $\overline{a+b}:a::\overline{c+d}:c$. (*M.* 1843, 1848, and 1849.)

(1) Art. 101. (2) Art. 102. (3) Art. 107. (4) The rectangle under the extremes is equal to the rectangle under the means. (5) Arts. 142 and 144.

(5) When are four quantities said to be proportional? If $a:b::c:d$, prove $a^2:b^2::c^2:d^2$. (*M.* 1844.)

(1) Art. 101. (2) Art. 147.

(6) Define ratio and proportion. (2) If $a:b::b:c$, prove $a:c::a^2:b^2$. (3) If $a\,x-b\,y=c\,y-d\,x$, find the ratio of x to y. (*M.* 1845 and 1848.)

(1) Arts. 100 and 101

(2) $\because \frac{a}{b}=\frac{b}{c}$, $\therefore \frac{a}{b}\times\frac{a}{b}=\frac{b}{c}\times\frac{a}{b}$, or $\frac{a^2}{b^2}=\frac{a\,b}{c\,b}$, $\therefore a:c::a^2:b^2$.

(3)
$$\because a\,x-b\,y=c\,y-d\,x$$
$$\therefore a\,x+d\,x=c\,y+b\,y$$
$$(a+d)\,x=(c+b)\,y$$
$$\frac{x}{y}=\frac{c+b}{a+d}, \text{ or } x:y::c+b:a+d.$$

VARIATION.

152. One quantity is said to *vary* as another when the two quantities depend upon each other in such a manner, that if the one be changed the other is changed in the same proportion.

Let A and B depend upon each other, in such a way, that if A be changed to any other value a, B must be changed to another value b; then A is said to *vary directly* as B, or $A : a :: B : b$.

The symbol of variation is $\propto$; and the expression A varies directly as B is indicated thus, $A \propto B$.

One quantity is said to *vary inversely* as another, when, the former being increased, the latter is diminished in a corresponding degree; or where the former being diminished the latter is increased in a similar manner.

A varies inversely as B when $A : B :: \frac{1}{a} : \frac{1}{b}$;

In symbols thus $A \propto \frac{1}{a}$.

In illustration of the foregoing remarks we may observe that if we double the base of a triangle, the vertex remaining the same, we know by *Euclid* VI., 1, that we double the area, and that in whatever proportion we alter the base the area is altered in the same proportion; hence we should say that (the altitude being given) the area of a triangle *varies directly as the base.*

153. Again, if A and a represent the altitude of two triangles, B and b their bases; then if the triangles have equal areas, $A \times B = a \times b$. Or, $A : a :: \frac{1}{B} : \frac{1}{b}$, that is, the altitudes of two equal triangles vary *inversely* as their bases.

One quantity is said to *vary as two others jointly*, if, when the former is changed in any manner, the product of the other two is changed in the same proportion.

Let S and s be the areas of two triangles whose altitudes are A a, and bases B, b, then $S : s :: A \times B : a \times b$.

Also if C and c be the contents of two solid parallelopipeds whose lengths are L, l, breadths B, b, and thicknesses T, t, then $C : c :: L \times B \times T : l \times b \times t$, where one quantity (the content) *varies as three others jointly.*

154. The relation expressed by $y \propto x$ is equivalent to the equation $y = C x$, where C is some constant quantity; for $\frac{y}{x}$ is the ratio of y to x, and the preceding equation expresses that this is constant, or always the same, whatever values x and y may have; and this is the same thing as saying that when one is increased the other is increased in the same proportion.

EXAMPLES.

(1) If $A \propto B$ and $B \propto C$: then will $A \propto C$;

$$\begin{aligned}&\text{For, if } A = p\,B \text{ and } B = q\,C,\\ &\text{We have } A = p\,B = p\,q\,C,\\ &\text{But } p\,q \text{ is constant } \therefore A \propto C.\end{aligned}$$

(2) If $A \propto \frac{1}{B}$, and $B \propto C$, then will $A \propto \frac{1}{C}$;

$$\begin{aligned}&\text{Let } A = \frac{p}{B} \text{ and } B = q\,C,\\ &\text{Then } A = \frac{p}{q\,C} = \frac{p}{q} \times \frac{1}{C};\\ &\text{But } \frac{p}{q} \text{ is constant } \therefore A \propto \frac{1}{C}.\end{aligned}$$

(3) If $A \propto \frac{1}{B}$ and $B \propto \frac{1}{C}$; then will $A \propto C$;

For if $A = \frac{p}{B}$ and $B = \frac{q}{C}$, then $A = \frac{p}{q}\,C$, or $A \propto C$.

(4) If S vary as A when B is given, and vary as B when A is given, generally it must vary as their product $A\,B$. ($B.\,A.$ 1838.)

$$\begin{aligned}&\text{For we may manifestly assume } S = p\,A\,B,\\ &\text{and } S = q\,B\,A,\\ &\text{whence } S^2 = p\,q\,(A.B)^2,\\ &S = \sqrt{p\,q}\,A.B.,\\ &\text{but } \sqrt{p\,q} \text{ is constant,}\\ &\therefore S \propto A.\,B.\end{aligned}$$

(5) If $A \propto B$ then will $A\,P \propto B\,P$, and $\frac{A}{P} \propto \frac{B}{P}$, where P may be either variable or invariable.

$$\begin{aligned}&\text{For if } A = p\,B, \text{ we have } A\,P = p\,B\,P,\\ &\text{and } \frac{A}{P} = p\,\frac{B}{P},\end{aligned}$$

whence it follows that, $A\,P \propto B\,P$ and $\frac{A}{P} \propto \frac{B}{P}$.

Hence also $A^m = p^m\,B^m$ and $A^m \propto B^m$, where m may be either integral or fractional.

(6) If $A \propto B$ when C is constant, and if $A \propto \frac{1}{C}$ when B is constant, prove that generally $A \propto \frac{B}{C}$. ($B.A.$ 1862.)

$$\text{Let } A = m\,B, \text{ and } A = \frac{m'}{C}$$

$\therefore$ A increases as B increases, and also as C diminishes, or A varies directly as B, and inversely as C.

$$\therefore A \propto \frac{B}{C}.$$

(7) If $x \propto y$, prove that $x^2 + y^2 \propto x\,y$. Also if $x^3 + \frac{1}{y^3} \propto x^3 - \frac{1}{y^3}$, prove that $x\,y$ is constant. ($B.A.$ 1862.)

(1) Let $x = m\,y$, then $x^2 = m^2 y^2$

$$x^2 + y^2 = (m^2 + 1)\,y^2$$

also $x\,y = m\,y^2$.

Since m is constant, and also m^2+1

∴ $x^2 + y^2$ and $x\,y$ both $\propto y^2$

or $x^2 + y^2 \propto x\,y$.

(2) If $x^3 + \frac{1}{y^3} \propto x^3 - \frac{1}{y^3}$

$$\therefore x^3 . y^3 + 1 \propto x^3 y^3 - 1$$

Let $x^3 y^3 + 1 = m\,(x^3 y^3 - 1)$

then $\overline{m - 1} . x^3 y^3 = m + 1$

$$\therefore x^3 y^3 = \frac{m+1}{m-1}$$

$$x\,y = \sqrt[3]{\frac{m+1}{m-1}} = \text{constant.}$$

(8) If A vary as B when C is constant, and vary as C when B is constant, prove that A will vary as B, C, when neither is constant. (*M.* 1861.)

See Qu. 4, and also :—Let $A = m\,B . C = m'\,C . B$

$$\therefore A^2 = m\,m'\,B^2 . C^2$$

$$A = \sqrt{m\,m}\;B . C$$

or $A \propto B . C$.

SIMPLE INTEREST.

155. Interest is the per centage given for the loan or use of money for a certain time.

The sum of money used or lent is called the *principal.*

The per centage is called the *rate.*

If the interest is supposed to be paid when it becomes due it is called *simple interest.* If it is added to the original sum, so as to increase the principal at the end of successive periods, it is called *compound interest.*

The period for which interest is given is called the *time.*

The interest added to the principal form the *amount.*

156. Questions in interest always furnish three of the quantities just defined, from which the fourth may be found. They are therefore divided into four classes—

Where the interest is required.—I.
,, principal ,, II.
,, time ,, III.
,, rate ,, IV.

I. Where the interest is required.

1. To find the interest for one year, multiply the principal by the rate and divide by 100, or $P . \frac{r}{100}$. Suppose it required to find the interest on £225 for one year, at 5 per cent. per annum.

$$\begin{array}{r} 225 \\ 5 \\ \hline 11{\cdot}25 \\ 20 \\ \hline 5{\cdot}00 \end{array}\quad \textit{Ans. } \pounds 11\ 5s.$$

2. Where the interest is required for several years, or for part of a year, the interest for one year must be multiplied or divided accordingly. It is better to multiply both by rate and time before dividing by 100.

The rule is expressed generally by the equation

$$I = \frac{P.r.t}{100},$$

where $I =$ the interest, P the principal, r the rate per cent., and t the number of periods for which r is payable.

Ex.—Find the simple interest on £587 10*s.* 6*d.*, for $7\frac{1}{2}$ years, at $3\frac{1}{2}$ per cent. per ann. (*M.* 1839.)

$$7\tfrac{1}{2} \times 3\tfrac{1}{2} = \frac{105}{4}.$$

$$\begin{array}{rrrl}
 & 587 & 10 & 6 \\
 & & & 5 \times 7 \times 3 = 105 \\
\hline
 & 2937 & 12 & 6 \\
 & & & 7 \\
\hline
 & 20563 & 7 & 6 \\
 & & & 3 \\
\hline
4) & 61690 & 2 & 6 \\
\hline
100) & 154{\cdot}22 & 10 & 7\tfrac{1}{2} \\
 & 20 & & \\
\hline
 & 4{\cdot}50 & & \\
 & 12 & & \\
\hline
 & 6{\cdot}07 & & \\
 & 4 & & \\
\hline
 & {\cdot}30 & & \textit{Ans.}\ £154\ 4s.\ 6\tfrac{3}{10}d.
\end{array}$$

157. II. To find the principal.

Solving the above equation with respect to P, we have

$$P = \frac{100\,I}{r\,t}.$$

Or, to find the principal, we multiply the interest by 100, and divide by the product of the rate per cent. and the number of periods.

Ex.—What principal will gain £200 in 8 years, at $2\frac{1}{2}$ per cent.?

$$P = \frac{200 \times 100}{8 \times 2\frac{1}{2}} = \frac{20000}{20} = £1000. \quad \textit{Ans.}$$

158. III. To find the time.

The above equation $I = \frac{P\,r\,t}{100}$ gives, $t = \frac{100\,I}{P\,r}$.

Ex.—In what time will £225 amount to £317, at 5 per cent. per annum? (*M.* 1840.)

$$317 - 225 = 92 = I.$$

$$t = \frac{92 \times 100}{225 \times 5} = \frac{368}{45} = 8\tfrac{8}{45} \text{ years.} \quad \textit{Ans.}$$

IV. To find the rate.

The equation $I = \frac{P\,r\,t}{100}$ gives $r = \frac{100\,I}{P\,t}$.

Ex.—At what rate per cent. will £300 amount to £373 10*s.* in 7 years? (*M.* 1862.)

$$r = \frac{73 \cdot 5 \times 100}{300 \times 7} = \frac{73 \cdot 5}{21} = 3 \cdot 5 = 3\tfrac{1}{2} \text{ per cent.} \quad \textit{Ans.}$$

EXAMPLES.

(1) What is the interest of £273 15*s.* for a year at 3¼ per cent.? (*B.A.* 1851.)

$$\text{(I.)} \quad P = \frac{273 \cdot 75 \times 3 \cdot 25}{100} = 8 \cdot 896875 = £8\ 17s.\ 11\tfrac{1}{4}d. \quad \textit{Ans.}$$

(2) At what rate per cent., simple interest, will £225 amount to £256 10*s.* in 4 years? (*B.A.* 1853.)

$$\text{(IV.)} \quad r = \frac{100 \times 31 \cdot 5}{225 \times 4} = 3\tfrac{1}{2}. \quad \textit{Ans.}$$

(3) In what time will £350 amount to £402 10*s.* at 3 per cent. simple interest? (*B.A.* 1854.)

$$\text{(III.)} \quad t = \frac{100 \times 52 \cdot 5}{350 \times 3} = 5 \text{ years.} \quad \textit{Ans.}$$

(4) If a sum of money doubles itself in 40 years at simple interest, what is the rate of interest? (*M.* 1858.)

Let the sum be £100: then by Art. 158, formula

$$\text{(IV.)} \quad r = \frac{100 \times 100}{100 \times 40} = 2\tfrac{1}{2} \text{ per cent.} \quad \textit{Ans.}$$

(5) In how many years will £625 10*s.* amount to £813 3*s.* at 4 per cent. simple interest? (*M.* 1859.)

$$\text{(III.)} \quad t = \frac{100 \times 187 \cdot 65}{625 \cdot 5 \times 4} = 7\tfrac{1}{2} \text{ years.} \quad \textit{Ans.}$$

(6) At what rate per cent., simple interest, will £7,433 6*s.* 8*d.* amount to £9,942 1*s.* 8*d.* in 7½ years? (*M.* 1860.)

$$\text{(IV.)} \quad r = \frac{100 \quad 2508 \cdot 75}{7433 \cdot 3 \times 7 \cdot 5} = 4\tfrac{1}{2} \text{ per cent.} \quad \textit{Ans.}$$

(7) What must be the rate per cent. that £73 may amount to £100 in 25 years? (*M.* 1860.)

$$\text{(IV.)} \quad r = \frac{100 \times 27}{73 \times 25} = 1\tfrac{35}{73} \text{ per cent. } \textit{Ans.}$$

(8) Find the rate per cent. at which £1,000 must be laid out at simple interest to become £1,100 in 5 years? (*B.A.* 1863.)

$$\text{(IV.)} \quad r = \frac{100 \times 100}{1000 \times 5} = 2 \text{ per cent. } \textit{Ans.}$$

DISCOUNT.

159. *Discount* is the abatement made for the payment of money before it becomes due.

The *present value* of any debt due some time hence, is such a sum as being put out to interest for that time will amount to the debt. Hence true discount is charged on any sum when the difference between the sum and its present value is the same as the interest on the present value; thus, £105 due at the end of a year, supposing interest at 5 per cent., is now worth £100, because £100 put out to interest, would, at the year's end, amount to £105. The present value and discount, therefore, upon £105, due one year hence, are respectively £100 and £5.

☞ In practice, bankers charge as discount, the interest for the given time upon the future debt, and therefore the sum obtained from them is always less than the true present value. Thus from £100, due a year hence, interest at 5 per cent., a banker takes £5 for discount, yet £95 put out to interest for one year, at 5 per cent., will not, with its interest, amount to £100.

160. Suppose the present value of £1,456 10*s.*, due 3 years hence, required, and interest at 4½ per cent. (*M.* 1847.)

If to the present value, were added 4½ per cent. of it every year, for 3 years, the sum ought to be £1,456 10*s.* exactly.

In 3 years, therefore, three times 4½, or 13½ per cent. of the present value will have been added to the present value to form the given amount, £1,456 10*s.*

It follows, then, supposing £1,456 10*s.* divided by 113½, that 100 times the quotient is the present value, and 13½ times the quotient is the interest that has been added to the present value to form the given amount. Hence, to find the present value of a sum of money,—

Multiply the sum by 100, and divide the product by 100 plus the rate per cent. multiplied by the number of years.

Or it may be stated thus:

£100 + its interest for the given time at the given rate per cent. : £100 : : given sum : *present value;* which in the question above becomes

£113½ : £100 : : £1456½ : *present value.*

$$\therefore \textit{Present value} = \frac{100 \times 1456{\cdot}5}{113{\cdot}5}$$

$\doteq$ £1283 5*s.* 2¼*d.* $\frac{117}{227}$

Discount = £1456 10*s.* — £1283 5*s.* 2¼*d.* $\frac{117}{227}$

= £173 4*s.* 9½*d.* $\frac{110}{227}$.

EXAMPLES.

(1) Find the present value of £793 due six months hence, interest being at 3½ per cent. (*M.* 1842.)

Here 6 months being half-a-year we have 3½ × ½ = 1¾ = £1 15*s.*

Then £101 15*s.* : £100 : : £793 : *present value.*

$$\therefore \textit{Present value} = \frac{100 \times 793}{101{\cdot}75} = £779\ 7s.\ 2\tfrac{1}{2}d.\ \tfrac{298}{407}.$$

(2) Find the present value of £1374 10*s.* at 4½ per cent. simple interest due 3 years hence. (*M.* 1848.)

113·5 : 100 : : 1374·5 : 1211·01322.

£1211 0*s.* 3*d.* *Ans.*

(3) Find the discount on £3540 10*s.* due 3 years hence, at 4½ per cent. (*M.* 1851.)

113·5 : 100 : : 3540·5 : 3119·3832.

Again £3540·5 — 3119·3832 = £421 2*s.* 4·03*d.* *Ans.*

(4) What is the present value of £2063 14*s.* due 6 months hence, interest being at 3 per cent.? (*B.A.* 1847.)

101·5 : 100 : : 2063·7 : 2033·20197.

£2033 4*s.* 0¼*d.* *Ans.*

(5) Find the present value of £911 13*s.* 3*d.* due 5 years hence, at 3 per cent. simple interest. (*M.* 1861.)

115 : 100 : : 911.6625 : 792·75.

£792 15*s.* *Ans*

STOCKS.

161. Stocks is the name given to the money borrowed by our own or any other government or trading company, at some given rate of interest which is settled at the time the money is lent.

Thus, if Government were to borrow £1,000,000 at 3 per cent., and A. had lent £1000 of this sum, A. would be said to have £1000 3 *per cent. stock,* and would receive a *Government Bond,* or document entitling him to receive the interest (viz. £30) upon this stock from year to year until Government chose to repay the principal.

The amount of debt owing by the Government is called the National Debt, or the Funds. The Funds represent the credit of the country, which is bound to pay whatever debts are contracted by its Government. The Government, however, reserves to itself the option of paying off the principal at any future time whatever; pledging itself nevertheless to pay the interest on it regularly at fixed periods in the mean time.

The interest is paid half-yearly, and the document, entitling the possessor to receive it may be sold just as any other kind of property.

162. If the 3 per cent. consols (consolidated annuities) be quoted in the money market at 97⅛, the meaning is, that for £97 2*s.* 6*d.* of money a person can purchase £100 stock, which will entitle him to half-yearly dividends from Government, at the rate of 3 per cent. per annum on the stock held by him.

From a variety of causes the price of stock is continually varying. A fundholder can at any time convert his stock into money, and it will depend upon the price at which he disposes of his stock, as compared with that at which he bought it, whether he will gain or lose by the transaction.

163. When the price of £100 stock is more than £100 in money, the stock is said to be at *a premium.*

When the price of £100 stock is £100 in money, the stock is said to be at *par.*

When the price of £100 stock is less than £100 in money, the stock is said to be at *a discount.*

Purchases and *Sales* of Stock are usually made through agents, called stock-brokers, at the rate of £⅛ or 2*s.* 6*d.* per cent. upon the stock bought or sold; so that, in practice, when stock is bought, every £100 stock costs the buyer £⅛ more than the market price of the stock: and when stock is sold, the seller gets £⅛ less for every £100 stock sold than the market price.

No general rule has been attempted for the solution of questions in stocks; they principally refer to the comparative value of various per centages, and are usually determined by the application of proportion. See Art. 107.

EXAMPLES.

(1) What sum will purchase £2400 in the 3 per cent. *consols* at 89½ per cent?

Let x = the sum required.

Then £100 stock : £2400 stock : : £89½ money : £x money,

$$\text{whence } x = \frac{2400 \times 89\frac{1}{2}}{100} = £2148.$$

that is, £2148 sterling will purchase £2400 of this stock, when it is at 89½ per cent.

(2) If I buy £1520 3 per cent. consols at 93¼, and pay ⅛ for brokerage, what does it cost me?

Every £100 stock costs me £(93½ + ⅛) or £93⅝, therefore if x equal the money required,

$$\text{£}100 \text{ stock} : \text{£}1520 \text{ stock} : : \text{£}93\tfrac{5}{8} : \text{£}x$$
$$\text{whence } x = \text{£}1419\ 6s.$$

(3) If I sell £1920 13*s.* 4*d.* in the 3½ per cents. at 98⅞, brokerage being £⅛ per cent., what sterling money shall I receive?

£100 stock realizes £(98⅞ − ⅛) = £98¾, therefore if x = the money required,

$$\text{£}100 \text{ stock} : \text{£}1920\tfrac{2}{3} \text{ stock} : : \text{£}98\tfrac{3}{4} : \text{£}x$$
$$\text{whence } x = \text{£}1896\ 13s.\ 2d.$$

(4) A person invests £3000 in the 3 per cents. when they are at 90⅛; what interest will he receive half-yearly?

Here 90⅛ sterling produce £3 yearly, or £1 10*s.* half-yearly; and therefore we have

$$\text{£}90\tfrac{1}{8} : \text{£}3000 : : \text{£}1\tfrac{1}{2} : \text{£}x$$
$$\therefore x = \text{£}49\ 17s.\ 9\tfrac{1}{4}d.\ \tfrac{257}{451},$$

his half-yearly dividend.

(5) A person has £5685 stock, the annual interest on which is reduced from 3½ to 3¼ per cent.: what does he lose in income by the reduction, and what is his income after it? (*M.* 1844.)

$$\text{Interest of £}5635 \text{ at } 3\tfrac{1}{2} \text{ per cent.} = \frac{5635 \times 3\frac{1}{2}}{100} = \text{£}197\ 4s.\ 6d.$$
$$\text{Interest of £}5635 \text{ at } 3\tfrac{1}{4} \text{ per cent.} = \frac{5635 \times 3\frac{1}{4}}{100} = \text{£}183\ 2s.\ 9d.$$
$$\text{Diff.} = \text{£}14\ 1s.\ 9d.$$

therefore he loses £14 1*s.* 9*d.*; and his income, after the reduction, is £183 2*s.* 9*d.*

(6) If the 3 per cent. stock be at 98, and the 3¼ per cent. stock be at 101, which stock is it most advantageous to buy? What income will £5000 invested in the 3 per cent. stock produce?

Here £98 : £100 : : £3 : £3·0612245 the rate per cent. for the money invested in the 3 per cents.

Also £101 : £100 : : £3·25 : £$3\cdot\dot{2}17\dot{8}$ the rate per cent. for the money invested in the 3¼ per cents.

Now as £$3\cdot\dot{2}17\dot{8}$ per cent. will realize a larger income than £3·0612245 per cent., it is more advantageous to buy the 3¼ per cent. stock.

Again, the interest of £5000 at 3·0612245 per cent.

$$= \frac{5000 \times 3\cdot0612245}{100} = \text{£}153\ 1s.\ 2\tfrac{1}{2}d.$$

the income produced by £5000 invested in the 3 per cents.

(7) How much stock can be purchased by the transfer of £2000

stock from the 3 per cents. at 90, to the 3½ per cents. at 96; and what change will be effected in income by it? (*B.A.* 1853.)

96 : 90 : : 2000 : 1875

∴ £1875, the quantity of stock that can be purchased in the 3½ per cents.

Also £2000 at 3 per cent. gives an income of £60, and £1875 at 3½ gives an income of £65 12*s.* 6*d.*; therefore the *increase* of income is £5 12*s.* 6*d.*

(8) The annual divisible receipts of a railway company are £437,500, and there are 250,000 shares at £21 each; what would be the dividend for each share, and what rate per cent. would be paid on each share? (*M.* 1852.)

$$\frac{£437,500}{250,000} = £1\ 15s.\ \text{dividend on each share.}$$

21 : 100 : : 1·75 : 8·3 = 8⅓ per cent. paid on each share.

(9) A man bought £500 3 per cent. stock, when consols were at 93½, and after having received one dividend, he sold at 96¾. What did he gain by the transaction? (*M.* 1860.)

100 : 96·75 : : 500 : 483·75 = £483 15*s.* received for stock.
100 : 93·5 : : 500 : 467·5 = £467 10*s.* original outlay.

£16 5*s.* gain by sale.

Dividend £15 + £16 5*s.* = £31 5*s.* gained by the transaction.

(10) The 3 per cents. being at 93, determine the interest obtained for money thus invested. (*B.A.* 1863.)

03 : 100 : : 3 : 3$\frac{7}{31}$ per cent.

CHAPTER VIII.

PERMUTATIONS AND COMBINATIONS, COMPOUND INTEREST, AND ANNUITIES FOR TERMS OF YEARS.

PERMUTATIONS AND COMBINATIONS.

164. The possible variations or arrangements or the *permutations* of any number of quantities, are the different ways in which they may be placed, *regarding their order.*

The possible selections or the *combinations* of any number of quantities are the different ways in which they may be taken, *disregarding their order.*

Permutations may be arranged two and two, three and three, four and four, &c., together: thus, of the four quantities a, b, c, d, the permutations two and two together are $a\,b$, $a\,c$, $a\,d$, $b\,a$, $b\,c$, $b\,d$, $c\,a$, $c\,b$, $c\,d$, $d\,a$, $d\,b$, $d\,c$; and the corresponding combinations, $a\,b$, $a\,c$, $a\,d$, $b\,c$, $b\,d$, $c\,d$.

PERMUTATIONS.

165. To find the number of permutations of m things arranged two and two together.

Let the quantities be a, b, c, d, &c. If we place one of the quantities, as a, we have $\overline{m-1}$ remaining, before each of which a may be placed; similarly of b, c, d, &c., and this operation may be performed m times.

$\therefore$ Permutations of m things taken two and two or $P_2 = m\,.\,\overline{m-1}$.

To find the permutations of m things arranged three and three together.

If one of the quantities, as a, be placed, the permutations of the remaining $\overline{m-1}$ quantities taken two and two are, as has just been shown, $\overline{m-1}\,.\,\overline{m-2}$, before each of which a may be placed. Similarly of b, c, d, &c., and this operation may be repeated m times.

$\therefore$ Permutations of three, or $P_3 = m\,.\,\overline{m-1}\,.\,\overline{m-2}$.

By similar reasoning, $P_4 = m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}$.

$P_5 = m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\,.\,\overline{m-4}$.

P_r, or permutations of m things arranged r together, $(r < m)$ $= m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,.\,.\,.\,.\,\overline{m-r+2}\,.\,\overline{m-r+1}$. . . . . . (I.)

To find the permutations of m things placed all together.

Make $m = r$ in the last expression: then

$$P_m = m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \overline{m-3} \ldots\ldots 3 \cdot 2 \cdot 1. \quad \text{(II.)}$$

Or take the product of all the natural numbers, up to and including the number of things permuted.

Ex.—In how many different ways can 7 persons be arranged on 7 different seats? (*B. A.* 1847.)

Here P_m is the permutations of 7 things taken 7 together,

$$= 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

Supposing two things given to form permutations of two with, the rule gives 2 . 1, or 2. But if the things are alike, only one permutation of two can be obtained. So if three things be given, the rule is 3 . 2 . 1, or 6; but if the three things given are all alike, only one permutation can be got. Similarly for four or more. In order, therefore, to obtain the true number of permutations in cases where things are thus repeated, the formulæ for permutations of 2, 3, and 4 of any given quantities must be divided by 2 . 1, 3 . 2 . 1, 4 . 3 . 2 . 1, respectively. If, for example, it were asked, How many permutations may be formed of the letters *a a a a b b b c c*? They are 9 in number, and, consequently, if they were all different, would admit of 9 . 8 . 7 . 6 . 5 . 4 . 3 . 2 . 1 permutations. But since a is found 4 times, this number must be divided by 4 . 3 . 2 . 1; and since b is found 3 times, it must also be divided by 3 . 2 . 1; and lastly, since c is found twice, it must be divided by 2 . 1; and therefore the number of permutations will be—

$$\frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 1260.$$

The general expression for the number of permutations that can be formed of m things, p being of one sort, and q of another, is—

$$P_m = \frac{1 \cdot 2 \cdot 3 \cdot \&c. \; m.}{(1 \cdot 2 \cdot \&c. \, p.)\,(1 \cdot 2 \cdot \&c. \; q.)} \quad \ldots\ldots\ldots\ldots \text{(III.)}$$

EXAMPLES.

(1) How many distinct permutations can be formed with the letters in the word *degree*? (*B. A.* 1845.)

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1} = 120. \; \textit{Ans.}$$

(2) How many different arrangements can be formed of the letters in the word *engine*? (*B. A.* 1851.)

$$\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot \; 2 \cdot 1} = 180. \; \textit{Ans.}$$

(3) From a company of 50 men, four are chosen every night to guard. (1) On how many different nights can a different guard be posted; and (2) on how many of these will any particular man be engaged? (*B. A.* 1850.)

$$(1)\ \frac{50\,.\,49\,.\,48\,.\,47\,.}{4\,.\,3\,.\,2\,.\,1} = 230300;\quad (2)\ \frac{230300}{50} = 4606.$$

(4) In how many ways may seven balls be arranged, in two divisions, so that the first division may contain three of the balls, the second four? (*B. A.* 1852.)

$$\frac{7\,.\,6\,.\,5\,.\,4\,.\,3\,.\,2\,.\,1\,.}{3\,.\,2\,.\,1\,.\,4\,.\,3\,.\,2\,.\,1} = 35.\ \textit{Ans.}$$

(5) How many words of four consonants and one vowel can be formed from seven consonants and three vowels, the vowel being always in the middle place. (*B. A.* 1863.)

$P_4 = 7\,.\,6\,.\,5\,.\,4 = 840$ for the consonants, and with one vowel always central,

$$= 840 \times 3 = 2520.\ \textit{Ans.}$$

COMBINATIONS.

166. The combinations of m things taken two and two together.

Every combination taken two and two together, as $a\,b$, may be permuted 2 . 1 different ways;

$$\therefore 2\,C_2 = P_2 \qquad \therefore C_2 = \frac{P_2}{2\,.\,1} = \frac{m\,.\,\overline{m-1}}{2\,.\,1}.$$

Combinations of m things taken three together.

Every such combination, as $a\,b\,c$, may be permuted 3 . 2 . 1 different ways.

$$\therefore 3\,.\,2\,.\,1\,C_3 = P_3 \quad \therefore C_3 = \frac{P_3}{3\,.\,2\,.\,1} = \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}}{3\,.\,2\,.\,1}.$$

Similarly we find—

$$C_4 = \frac{P_4}{4\,.\,3\,.\,2\,.\,1} = \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}}{4\,.\,3\,.\,2\,.\,1};$$

$$\text{and generally, } C_r = \frac{P_r}{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,\ldots\,3\,.\,2\,.\,1},$$

$$= \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,\ldots\,\overline{m-r+2}\,.\,\overline{m-r+1}}{r\,.\,\overline{r-1}\,.\,\overline{r-2}\,\ldots\,.\,3\,.\,2\,.\,1}\ \ldots\ (\text{I.})$$

$$C_m = \frac{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\,.\,\overline{m-4}\,\ldots\ldots\,3\,.\,2\,.\,1}{m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\,.\,\overline{m-4}\,\ldots\ldots\,3\,.\,2\,.\,1}.\ (\text{II.})$$

Ex.—How many combinations of 6 can be made out of 8 things?

$$\textit{Ans. } \frac{8 \,.\, 7 \,.\, 6 \,.\, 5 \,.\, 4 \,.\, 3}{6 \,.\, 5 \,.\, 4 \,.\, 3 \,.\, 2 \,.\, 1} = 28.$$

To show that the number of combinations of m things, taken r and r together, is equal to the number of combinations of m things, taken $m - r$ and $m - r$ together.

If in the expression above, for C_r we substitute $m - r$ for r, we shall have—

$$C_{m-r} = \frac{m\,(m-1)\,(m-2)\ \&c.\ (r+1)}{1 \,.\, 2 \,.\, 3\ \&c.\ (m-r)}$$

$$= \frac{m\,(m-1)\,(m-2)\ \&c.\ (r+1) \,.\, r \,.\, \&c.\ 3 \,.\, 2 \,.\, 1}{1 \,.\, 2 \,.\, 3\ \&c.\ r \,.\, 1 \,.\, 2 \,.\, 3\ \&c.\ (m-r)}$$

$$= \frac{m\,(m-1)\,(m-2)\ \&c.\ (m-r+1)}{1 \,.\, 2 \,.\, \&c.\ r}, \times \frac{(m-r)\ \&c.\ 3 \,.\, 2 \,.\, 1}{1 \,.\, 2 \,.\, 3\ \&c.\ (m-r)}$$

$$= \frac{m\,(m-1)\,(m-2)\ \&c.\ (m-r+1)}{1 \,.\, 2 \,.\, 3\ \&c.\ r} = C_r \ldots \text{(III.)}$$

Another method of proving this proposition is as follows:—Whenever r of the m quantities are taken to form a combination, $m - r$ are necessarily omitted, and these may be supposed to be formed into a combination which may be called, with reference to the former, a *complementary* combination. Hence each combination of r quantities has its complementary combination of $m - r$, and therefore the number of the two sets of combinations is equal.

Ex.—The number of combinations of five (m) things, as a, b, c, d, e taken three (r) and three (r) together, is equal to the number of combinations of five things taken two ($m - r$) and two together.

Taken three and three, we have $C_3 = \frac{m \,.\, \overline{m-1} \,.\, \overline{m-2}}{1 \,.\, 2 \,.\, 3}$ $= \frac{5 \,.\, 4 \,.\, 3}{1 \,.\, 2 \,.\, 3} = 10$ combinations, *viz.*:—

$$\overset{1}{abc},\ \overset{2}{abd},\ \overset{3}{abe},\ \overset{4}{acd},\ \overset{5}{ace},\ \overset{6}{ade},\ \overset{7}{bcd},\ \overset{8}{bce},\ \overset{9}{bde},\ \overset{10}{cde}.$$

Taken two and two, we have $C_2 = \frac{m \,.\, \overline{m-1}}{1 \,.\, 2} = \frac{5 \,.\, 4}{1 \,.\, 2} = 10$ combinations, *viz.*:—

$$\overset{10}{ab},\ \overset{9}{ac},\ \overset{8}{ad},\ \overset{7}{ae},\ \overset{6}{bc},\ \overset{5}{bd},\ \overset{4}{be},\ \overset{3}{cd},\ \overset{2}{ce},\ \overset{1}{de}.$$

The numbers point out the combinations that are complementary to each other. Thus $c\,d\,e$ has $a\,b$ for its complementary combination.

COMPOUND INTEREST.

167. In compound interest the amount at the end of the first period is put to interest, and at the end of the second period the interest on this amount is added to it, and the interest on this second amount is then taken, and so on, till the number of periods required is completed.

Hence the rule: Find the amount of the given principal for the time of the first period by simple interest, then consider this amount as the principal for the second period, and find its amount as before. Proceed thus through all the periods, always considering the last amount as the principal for the next period; then, if the given principal be deducted from the last amount, the remainder will be the interest required.

Ex.—Required the compound interest on £1000 for 5 years, at 2½ per cent. per annum.

1000	. . .		1st	year's	principal
25	. . .			,,	interest
1025	. . .			,,	amount = 2nd year's prin.
25	12	6 .	2nd	,,	int.
1050	12	6 .		,,	amt. = 3rd ,, ,,
26	5	3¾.	3rd	,,	int.
1076	17	9¾.		,,	amt. = 4th ,, ,,
26	18	5¼.	4th	,,	int.
1103	16	3 .		,,	amt. = 5th ,, ,,
27	11	10¾.	5th	,,	int.
1131	8	1¾.		,,	amt.
1000	0	0			

131 8 1¾ = whole interest: which is £6 8*s.* 1¾*d.* more than simple interest of the same sum.

This process is the same that would be employed in finding the compound interest on £1000 for 2½ years, at 5 per cent. per annum, payable half-yearly.

But taking the expression for the simple interest for one year (Art. 159), we see that the amount for the 1st year is $p + p\frac{R}{100}$, or $p\left(1 + \frac{R}{100}\right)$ where $\frac{R}{100}$ is the interest of one pound for one year; and, therefore, $1 + \frac{R}{100}$ is the amount of one pound for one year. Hence to obtain the amount of any given sum at interest for one year, we may multiply the principal by the amount of one pound for one year at the given rate. The amount, at compound interest, of the second year, may be found in precisely the same manner; so that if A_n mean the amount at the end of the *n*th year,

$$A_1 = p\left(1 + \frac{R}{100}\right)$$

$$A_2 = p\left(1 + \frac{R}{100}\right)\cdot\left(1 + \frac{R}{100}\right) = p\left(1 + \frac{R}{100}\right)^2$$

$$A_3 = p\left(1 + \frac{R}{100}\right)^2\cdot\left(1 + \frac{R}{100}\right) = p\left(1 + \frac{R}{100}\right)^3$$

$$A_n = p\left(1 + \frac{R}{100}\right)^{n-1}\cdot\left(1 + \frac{R}{100}\right) = p\left(1 + \frac{R}{100}\right)^n$$

Hence the following rule:—

To find the amount of any sum at compound interest for any number of periods:

Involve the amount of £1, at the given rate, for the first period, to the power expressed by the number of periods, and multiply the result by the given principal.

EXAMPLES.

(1) £1000 for 5 years, at 2½ per cent. per annum, compound interest.

Amount of £1 for one year, at 2½ per cent. 1·025.

$$\begin{aligned} (1{\cdot}025)^5 &= 1{\cdot}131408 \\ &\quad\ \ 1000 \\ \hline &\ 1131{\cdot}408 = £1131\ 8s.\ 2d.\ \textit{Ans.} \end{aligned}$$

(2) What will £650 amount to in 5 years, at 5 per cent. compound interest? (*B. A.* 1846.)

Amount of £1 for one year, at 5 per cent. = 1·05.

$$\begin{aligned} (1{\cdot}05)^5 &= 1{\cdot}2762816 \\ &\quad\ \ 650 \\ \hline &\ 829{\cdot}58304 = £829\ 11s.\ 7\tfrac{3}{4}d. \end{aligned}$$

(3) In what time will a sum of money double itself, at 3½ per cent. per annum compound interest? (*B. A.* 1841.)

Log. 2 = ·3010300

Log. 1·035 = ·0149403

$$\text{Here } A_n = p\left(1 + \frac{R}{100}\right)^n = 2\,p \text{ by the question.}$$

$$\therefore \left(1 + \frac{R}{100}\right)^n = 2.$$

$$\text{But } 1 + \frac{R}{100} = 1{\cdot}035.$$

$$\therefore (1{\cdot}035)^n = 2.$$

$$n \,.\, \log.\ 1{\cdot}035 = \log.\ 2.$$

$$\text{Whence } n = \frac{\log.\ 2}{\log.\ 1{\cdot}035} = \frac{{\cdot}3010300}{{\cdot}0149403} = 20{\cdot}1488597 \text{ years.}$$

ANNUITIES.

168. An annuity is a sum of money receivable at certain fixed periods.

Annuities are either immediate or deferred. The first payment of an immediate annuity is to be made a year hence. The first payment of an annuity deferred, say for 10 years, is to be made 11 years hence.

169. The present value of an annuity is of course the sum of the present values of each of the payments. Thus the present value of an immediate annuity of £100 for 5 years is

The present value of £100 due 1 year hence
+ Do. do. 2 years hence
+ Do. do. 3 „
+ Do. do. 4 „
+ Do. do. 5 „

Now the present value of £100 due 1 year hence is (Art. 163) $\frac{100 \times 100}{100 + R}$, or if $\frac{R}{100}$ (which is the interest of £1 for 1 year) $= r$

$$\frac{100 \times 100}{100 + R} = \frac{100}{1 + r}.$$

And this sum deferred one year (which is the same as £100 deferred for 2 years) $= \frac{100}{1 + r} \times \frac{1}{1 + r}$ or $\frac{100}{(1 + r)^2}$. And so on, for the remaining payments. The value of the annuity will therefore be

$$\frac{100}{1 + r} + \frac{100}{(1 + r)^2} + \frac{100}{(1 + r)^3} + \frac{100}{(1 + r)^4} + \frac{100}{(1 + r)^5},$$

which is evidently a geometrical series of which the first term is $\frac{100}{1 + r}$, the ratio $\frac{1}{1 + r}$, and the number of terms 5; the sum (Art. 117) is

$$\frac{100}{1+r}\,\frac{\left(\frac{1}{1+r}\right)^5 - 1}{\frac{1}{1+r} - 1}.$$

Multiplying the two denominators together, we have

$$100\,\frac{\left(\frac{1}{1+r}\right)^5 - 1}{-r},\ \text{or}\ 100\,\frac{1 - \left(\frac{1}{1+r}\right)^5}{r}.$$

Or taking any number of years n, we have the general expression for the present value of an immediate annuity of p pounds,

$$p\,\frac{1 - \left(\frac{1}{1+r}\right)^n}{r},$$

which is the algebraic form of the following rule:—

170. Find the present value of £1 due at the end of the first period: involve it to the power indicated by the number of periods. Subtract this result from unity; divide the remainder by the interest of £1 for one of the periods; and multiply the quotient by the periodical payment. This last product is the present value of the annuity.

Ex.—The present value of an immediate annuity of £200 for 10 years, at 5 per cent.

Present value of £1 due 1 year hence is $\frac{1}{1{\cdot}05}$

$\left(\frac{1}{1{\cdot}05}\right)^{10} = \frac{1}{1{\cdot}628895}$. $\quad 1 - \left(\frac{1}{1{\cdot}05}\right)^{10} = \frac{{\cdot}628895}{1{\cdot}628895}$; which $\div\ {\cdot}05 = \frac{12{\cdot}5779}{1{\cdot}628895}$ = present value of an annuity of £1 for 10 years.

$\frac{12{\cdot}5779}{1{\cdot}628895} \times 200 = \frac{2515{\cdot}58}{1{\cdot}628895} = 1544{\cdot}34755$ = £1544 6*s*. 11¼*d*. *Ans.*

171. If the annuity be deferred, this value must be multiplied by the present value of £1, due at the expiration of the term of deferment.

Thus, if the above annuity instead of being immediate were deferred for 20 years

Value of £1 due at the end of 20 years is ·3768895,

And $1544{\cdot}3428 \times {\cdot}3768895 = 582{\cdot}0467$,

Or the present value of the deferred annuity = £582 0*s*. 11*d*. *Ans.*

172. To find the annuity which a given sum will purchase for a given time at a given rate per cent.

From the rules just given, we see that the quotient of the given sum divided by the value of an annuity of £1 for the given time, is the annuity required.

EXAMPLES.

(1) What immediate annuity for 10 years will £1500 purchase, interest being reckoned at 5 per cent.?

Present value of an annuity of £1 for 10 years at 5 per cent. is as above, $\frac{12\cdot5779}{1\cdot628895}$, or 7·721737,

and $\frac{1500}{7\cdot721737} = 194\cdot25681 =$ £194 5*s.* 1¾*d.* *Ans.*

(2) What annuity for 10 years deferred for 20 years will £600 purchase, interest being reckoned at 5 per cent. ?

Value of immediate annuity of £1 for 10 years, at 5 per cent., is 7·721737.

Value of ditto deferred for 20 years is 7·721737 × ·3768895, or 2·910231.

Then $\frac{600}{2\cdot910231} = 206\cdot1692$, or £206 3*s.* 4½*d.* *Ans.*

In practice, these calculations are greatly abridged by the use of Tables, which show the value of £1 under different forms of accumulation, at different rates per cent.—See *Jones' Annuities, L.U.K.*

CHAPTER IX.

QUADRATIC EQUATIONS, AND QUESTIONS PRODUCING THEM.

173. A QUADRATIC equation, or an equation of the second degree, is one which contains the square, but no higher power of the unknown quantity.

174. A quadratic equation to one unknown quantity may consist of two terms only, the one involving the square of the unknown quantity, x^2, the other composed of quantities which are known. Or it may consist of three terms, the first involving the square of the unknown quantity x^2, the second involving the first power of the unknown quantity x, and the third composed of quantities which are known. Thus $x^2 = p$ is a pure quadratic equation; $x^2 + bx = c$ is an adfected quadratic equation. The former is also called an *incomplete*, and the latter a *complete* quadratic.

PURE QUADRATIC EQUATIONS.

175. These are of the form $x^2 = p$. Taking the square root of both sides, we have at once,

$$x = \pm \sqrt{p}.$$

The signs + and − are both prefixed to the root, because the square root of a quantity may be either positive or negative.

If the height, through which a heavy body falls in different periods of time be proportioned to the squares of those times, in how many seconds will a body fall through 1000 feet, the space it falls through in one second being 16·1 feet?

Let x = the number of seconds required:—

$$\text{Then } 16{\cdot}1 : 1000 :: (1'')^2 : (x''),$$

$$\text{whence } 16{\cdot}1\, x^2 = 1000,$$

$$x^2 = \frac{1000}{16{\cdot}1} = 62{\cdot}11,$$

$$x = \pm\, 7{\cdot}9 \text{ nearly.}$$

The negative value implies, that a heavy body thrown upwards with such a velocity as would make it ascend 1000 feet, would take 7·9 seconds to reach that height.

EXAMPLES.

(1) Given $51\,x^2 - 96 = 39\,x^2 + 96$, to find x.

$$\begin{aligned} \text{Here } 51\,x^2 - 39\,x^2 &= 192: \\ \text{That is, } 12\,x^2 &= 192; \\ x^2 &= 16; \\ x &= \pm 4. \end{aligned}$$

(2) Given $\left.\begin{array}{l} \frac{3}{x} + \frac{3}{y} = 6. \\ x + y = 2. \end{array}\right\}$ to find x and y. (*M.* 1838.)

$$\begin{aligned} \text{From (1) } x + y &= 2\,x\,y, \\ \text{(2) } x + y &= 2; \\ \therefore\ 2\,x\,y = 2 &\text{ and } x\,y = 1. \\ \text{Squaring (2), } x^2 + 2\,x\,y + y^2 &= 4 \\ 4\,x\,y &= 4 \\ \hline x^2 - 2\,x\,y + y^2 &= 0 \\ x - y = 0, &\text{ and } x + y = 2. \\ \therefore\ x = 1&, y = 1. \end{aligned}$$

(3) Given $\left.\begin{array}{l} x + y = 36 \\ x^2 + y^2 = 650 \end{array}\right\}$ to find x and y.

Squaring the first equation we have $x^2 + 2\,x\,y + y^2 = 1296$

Subtracting second $x^2 \qquad + y^2 = \ \ 650$

We have $\qquad 2\,x\,y \qquad = \ \ 646$

Subtracting this from the 2nd, we have $x^2 - 2\,x\,y + y^2 = 4$;

$$\begin{aligned} \therefore\ x - y &= \pm 2. \\ \text{But } x + y &= 36; \\ \therefore\ x = 19 \text{ or } 17;\ &\text{and } y = 17 \text{ or } 19. \end{aligned}$$

(4) Given $\left.\begin{array}{l} x^2 + x\,y = 40 \\ x\,y + y^2 = 60 \end{array}\right\}$ to find x and y.

Adding the two equations together—

$$\begin{aligned} x^2 + 2\,x\,y + y^2 &= 100; \\ \text{Whence } x + y &= \pm 10. \end{aligned}$$

But in the first equation $x\,(x + y) = 40$.

Substituting $\therefore\ \pm 10\,x = 40$ and $x = \pm 4$,

And $y = \pm 10 - x = \pm 6$.

(5) Given $\left.\begin{array}{l} x + y = a \\ x^2 + y^2 = b \end{array}\right\}$ to find x and y.

Squaring the first equation $x^2 + 2\,x\,y + y^2 = a^2$

But $x^2 \qquad + y^2 = b$

By subtraction $\qquad 2\,x\,y = a^2 - b$

$$\therefore x^2 - 2xy + y^2 = 2b - a^2;$$

$$\text{Whence } x - y = \pm\sqrt{2b - a^2} \;.\;.\;.\;.\;.\;.\;.\; (1)$$

$$\text{But } x + y = a \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (2)$$

$$\text{Add (1) and (2) } \therefore x = \frac{a \pm \sqrt{2b - a^2}}{2}$$

$$\text{Subtract (1) from (2) } \therefore y = \frac{a \mp \sqrt{2b - a^2}}{2}.$$

(6) Given $\left.\begin{aligned} x + y &= 18 \\ x^3 + y^3 &= 4914 \end{aligned}\right\}$ to find x and y. (*M.* 1838.)

Dividing the second equation by the first—

$$\begin{aligned} x^2 - xy + y^2 &= 273 \\ \text{Squaring the first } x^2 + 2xy + y^2 &= 324 \end{aligned}$$

$$\text{By subtraction } 3xy = 51 \text{ and } xy = 17;$$

$$\therefore x^2 - 2xy + y^2 = 256;$$

$$\text{Whence } x - y = \pm 16.$$

$$\text{But } x + y = 18;$$

$$\therefore x = \frac{18 \pm 16}{2} = 17 \text{ or } 1,$$

$$y = \frac{18 \mp 16}{2} = 1 \text{ or } 17.$$

(7) Given $\left.\begin{aligned} x^4y^3 - x^3y^4 &= 216 \\ x^2y - xy^2 &= 6 \end{aligned}\right\}$ to find x and y.

Dividing the first equation by the second—

$$x^2y^2 = 36 \text{ and } xy = \pm 6.$$

The second equation is $xy(x - y) = 6$,

$$\therefore 6(x - y) = 6,$$

$$\text{Whence } x - y = 1.$$

$$\begin{aligned} \text{Squaring } x^2 - 2xy + y^2 &= 1 \\ +4xy &= 24 \\ \hline \text{Adding } (x + y)^2 &= 25 \\ \therefore x + y &= 5 \\ \text{But } x - y &= 1 \\ \hline \therefore x = 3 \qquad y &= 2 \end{aligned}$$

(8) Given $\left.\begin{aligned} \sqrt{x} + \sqrt{y} : \sqrt{x} - \sqrt{y} :: 11 &: 5 \\ x + y &= 73 \end{aligned}\right\}$ to find x and y.

In the first equation multiplying extremes and means—

$$3\sqrt{x} = 8\sqrt{y} \text{ and } 9x = 64y.$$

$$\therefore x = \frac{64}{9}y.$$

Substitute this value of x in 2nd equation—

$$\text{Whence } \frac{64}{9} y + y = 73,$$

$$\text{And } y = 9;$$

$$\therefore x = 73 - y = 73 - 9 = 64.$$

(9) Find two numbers in the ratio of 5 : 3, the difference of whose squares shall be 144.

Let x and y = the numbers—

$$\left.\begin{aligned} \therefore x : y :: 5 : 3 \\ x^2 - y^2 = 144 \end{aligned}\right\} \text{by the question.}$$

From the first equation—

$$3x = 5y \text{ and } x = \frac{5}{3} y.$$

Substituting in the second equation—

$$\frac{25}{9} y^2 - y^2 = 144 \text{ or } \frac{16}{9} y^2 = 144,$$

$$\text{Whence } \frac{4}{3} y = 12,$$

$$y = 9,$$

$$\text{And } x = \frac{5}{3} y = 15.$$

Or assume $5x$ and $3x$ to be the numbers, then $25x^2 - 9x^2 = 144$, &c., produce the same answer as before.

(10) Divide 89 into two such parts that the sum of their square roots may be 13.

Let x^2 and y^2 = the two parts—

$$\left.\begin{aligned} \therefore x^2 + y^2 &= 89 \\ x + y &= 13 \end{aligned}\right\} \text{by the question.}$$

Squaring the second equation—

$$\begin{array}{rrcl} & x^2 + 2xy + y^2 &=& 169 \\ \text{But } & x^2 \qquad\quad + y^2 &=& 89 \\ \hline & 2xy \qquad\qquad &=& 80 \end{array}$$

$$\text{Whence } x^2 - 2xy + y^2 = 9,$$

$$\text{And } x - y = 3;$$

$$\text{But } x + y = 13,$$

$$\therefore x = 8 \text{ and } x^2 = 64,$$

$$y = 5 \text{ and } y^2 = 25.$$

Or suppose x, and $13 - x$, the roots, then $169 - 26x + 2x^2 = 89$, &c., give 64 and 25, as before.

ADFECTED QUADRATIC EQUATIONS.

176. Of these the general form is—

$$a x^2 + b x + c = 0.$$

To disengage x^2 from its coefficient, divide all the terms of the equation by a; then

$$x^2 + \frac{b}{a} x + \frac{c}{a} = 0.$$

If the term in x^2 has the sign $-$, the signs of all the terms must be changed, so that x^2 may always have the sign $+$;

$$\text{Let } \frac{b}{a} = p; \text{ and } \frac{c}{a} = q;$$

then $x^2 + p x + q = 0$, the general form to which equations of the second degree are reduced.

177. In order to solve the equation $x^2 + p x + q = 0$, we shall further reduce it to the form $(x + m)^2 = n$.

For this purpose let q be transposed to the second member of the equation, then

$$x^2 + p x = - q \text{ . . . . I.}$$

Now, since the square of a binomial is composed of (Art. 27) the square of the first term, + twice the product of the first and second terms, + the square of the second term, it follows that $x^2 + p x$ may be regarded as the first and second terms of $(x + \frac{1}{2} p)^2$. The first member of the above equation can therefore be made the square of $(x + \frac{1}{2} p)$, by adding to it the square of $\frac{1}{2} p$, that is $\frac{1}{4} p^2$. Adding this quantity to both sides of I., we have—

$$x^2 + p x + \frac{1}{4} p^2 = \frac{1}{4} p^2 - q \text{ . . . . II.}$$

$$\text{or} \left(x + \frac{1}{2} p\right)^2 = \frac{1}{4} p^2 - q.$$

Taking the square root—

$$x + \frac{1}{2} p = \pm \sqrt{\frac{1}{4} p^2 - q}.$$

$$\therefore x = - \frac{1}{2} p \pm \sqrt{\frac{1}{4} p^2 - q} \text{ . . . . III.}$$

178. From the equations marked I., II., and III., the process most generally used for the solution of a quadratic is derived.

I. presents the form to which every equation must be reduced, previous to applying the rule.

II. and III. give the algebraic explanation of the following rule:

Add the square of half the coefficient of the second term to both sides

of the equation. Extract the square root of each side, and from the resulting simple equation determine the value of x. •

⁂ The following principles are important in the application of the above rule to the actual solution of questions.

179. From the values just found, viz.:—

$$x = -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q},$$

$$x = -\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}.$$

180. If q be greater than $\frac{1}{4}p^2$, then $\sqrt{\frac{1}{4}p^2 - q}$ will be impossible, being the square root of a negative quantity; so that, if one value be impossible, the other will be impossible also: from which it appears that *imaginary* roots enter quadratic equations in pairs.

Ex.—Divide 20 into two such parts, that their product may be 101.

Let x and $(20 - x)$ be the parts,

$$\text{Then } x(20 - x) = 101 \text{ by the question.}$$
$$20x - x^2 = 101,$$
$$x^2 - 20x = -101,$$
$$x^2 - 20x + 100 = 100 - 101 = -1,$$
$$x - 10 = \pm\sqrt{-1},$$
$$x = 10 \pm\sqrt{-1},$$

which shows that it is impossible to comply with the conditions of the question, or that 20 cannot be divided into two such parts that their product shall be 101: indeed, the greatest product that can be made of the parts of a number, is obtained when they are equal to each other.

181. Also if q be equal to $\frac{1}{4}p^2$, the values of x will be equal.

Ex.—Let it be required to divide the number 20 into two such parts, that their product shall be 100.

$$\text{Here we have } x(20 - x) = 100,$$
$$x^2 - 20x = -100;$$
$$\text{Whence } x = 10 \pm\sqrt{100 - 100},$$
$$x = 10 \pm 0,$$

and the values of x are equal.

182. To show the formation of equations from given roots, we take the following example:—

Given $x^2 - 10x + 24 = 0$ to find the values of x.

$$\text{Here we have } x^2 - 10x = -24.$$

Completing the square by adding to both members 25, the square of half the coefficient of the second term, we obtain —

$$x^2 - 10\,x + 25 = 1.$$

Extracting the square roots of both sides, we have —

$$x - 5 = \pm 1,$$

$$\text{And } x = 6 \text{ and } 4.$$

Now if we wish to form the equation of which the roots are 6 and 4, we shall have $x - 6 = 0$, and $x - 4 = 0$;

$$\therefore (x - 6)(x - 4) = 0,$$

$$\text{Or } x^2 - 10\,x + 24 = 0, \text{ the original equation.}$$

Again, if we denote roots of a quadratic equation by α and β, we have,

$$\text{Since } x - \alpha = 0, \text{ and } x - \beta = 0,$$

$$(x - \alpha)(x - \beta) = 0,$$

$$\text{Or } x^2 - (\alpha + \beta)\,x + \alpha\,\beta = 0.$$

Whence it appears that the coefficient of the second term, with its sign changed, is equal to the sum of the roots, and the third term is equal to their product.

183. Every equation of the form,

$$x^{2m} + p\,x^m = q;$$

in which the index of x in the first term is double of its index in the second, may be solved by completing the square, m being either positive or negative, integral or fractional; thus, completing the square,

$$x^{2m} + p\,x^m + \frac{1}{4}p^2 = q + \frac{1}{4}p^2,$$

$$\therefore x^m + \frac{1}{2}p = \pm\sqrt{q + \frac{1}{4}p^2},$$

$$x^m = -\frac{1}{2}p \pm \sqrt{q + \frac{1}{4}p^2},$$

$$x = \left\{-\frac{1}{2}p \pm \sqrt{q + \frac{1}{4}p^2}\right\}^{\frac{1}{m}}.$$

EXAMPLES

(1) Given $x^4 + x^2 = 90$, to find x.

Completing the square, $x^4 + x^2 + \frac{1}{4} = \frac{361}{4}$,

$$\therefore x^2 + \frac{1}{2} = \frac{\pm 19}{2},$$

$$x^2 = 9 \text{ or } -10,$$

$$x = \pm 3 \text{ or } \pm\sqrt{-10}.$$

(2) Given $x^6 + 10\,x^3 = 144$, to find x.

$$\therefore\ x^6 + 10\,x^3 + 25 = 169.$$
$$x^3 + 5 = \pm 13,$$
$$\therefore\ x^3 = 8 \text{ or } -18.$$

Each of the equations, $x^3 = 8$, or $x^3 = -18$, will have three roots. Taking first $x^3 = 8$, or $x^3 - 8 = 0$, we have plainly one value, $x = 2$, or $x - 2 = 0$. Dividing $x^3 - 8 = 0$, by $x - 2 = 0$, we get the quadratic $x^2 + 2\,x + 4 = 0$, of which the roots are $x = -1 \pm \sqrt{-3}$; therefore, the roots of the equation $x^3 = 8$ are,

$$x = 2,$$
$$x = -1 + \sqrt{-3},$$
$$x = -1 - \sqrt{-3}.$$

Of $x^3 = -18$ or $x^3 + 18 = 0$, one root is $x = -2{\cdot}62074$, or $x + 2{\cdot}62074 = 0$. Dividing $x^3 + 18 = 0$ by this last equation, we obtain the quadratic, $x^2 - 2{\cdot}62074\,x + 6{\cdot}86826 = 0$; of which the roots are, $x = 1{\cdot}31037 \pm \sqrt{-5{\cdot}15120}$; therefore, the roots of the equation $x^3 = -18$ are,

$$x = -2{\cdot}62074,$$
$$x = 1{\cdot}31037 + \sqrt{-5{\cdot}15120},$$
$$x = 1{\cdot}31037 - \sqrt{-5{\cdot}15120}.$$

(3) Given $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$, to find x.

$$\text{Completing the square, } x^{\frac{6}{5}} + x^{\frac{3}{5}} + \frac{1}{4} = \frac{3025}{4}.$$

$$\text{Extracting the root, } x^{\frac{3}{5}} + \frac{1}{2} = \pm\,\frac{55}{2}.$$

$$\therefore\ x^{\frac{3}{5}} = 27 \text{ or } -28,$$

$$\text{Whence } x^{\frac{1}{5}} = 3 \text{ or } \sqrt[3]{-28},$$

$$x = 243 \text{ or } (-28)^{\frac{5}{3}}.$$

(4) Given $3\,x^{\frac{4n}{5}} - 4\,x^{\frac{2n}{5}} = 4$, to find x.

$$\text{Dividing by 3, } x^{\frac{4n}{5}} - \frac{4}{3}\,x^{\frac{2n}{5}} = \frac{4}{3}.$$

$$\text{Completing the square, } x^{\frac{4n}{5}} - \frac{4}{3}\,x^{\frac{2n}{5}} + \frac{4}{9} = \frac{16}{9}.$$

$$\text{Extracting the root, } x^{\frac{2n}{5}} - \frac{2}{3} = \pm\,\frac{4}{3}.$$

$$\text{Whence } x^{\frac{2n}{5}} = 2 \text{ or } -\frac{2}{3},$$

$$x^{2n} = 32 \text{ or } -\frac{32}{243},$$

$$x = (32)^{\frac{1}{2n}} \text{ or } \left(-\frac{32}{243}\right)^{\frac{1}{2n}}.$$

(5) Given $x^{-2} - 2x^{-1} = 8$, to find x.

Completing the square, $x^{-2} - 2x^{-1} + 1 = 9$,

$$\therefore x^{-1} - 1 = \pm 3,$$
$$x^{-1} = 4 \text{ or } -2,$$
$$\text{Or } \frac{1}{x} = 4 \text{ or } -2,$$
$$\text{Whence } x = \frac{1}{4} \text{ or } -\frac{1}{2}.$$

(6) Given $x^2 = 21 + (x^2 - 9)^{\frac{1}{2}}$, to find x.

Transposing, $(x^2 - 9) - (x^2 - 9)^{\frac{1}{2}} = 12$, a quadratic,

$$\text{Completing, \&c., } (x^2 - 9) - (x^2 - 9)^{\frac{1}{2}} + \frac{1}{4} = \frac{49}{4};$$
$$\text{Whence } (x^2 - 9)^{\frac{1}{2}} - \frac{1}{2} = \pm \frac{7}{2},$$
$$(x^2 - 9)^{\frac{1}{2}} = 4 \text{ or } -3,$$
$$x^2 - 9 = 16 \text{ or } 9,$$
$$x^2 = 25 \text{ or } 18,$$
$$x = \pm 5 \text{ or } \pm 3\sqrt{2}.$$

(7) Given $x^2 - 2x + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 11$, to find x.

Adding 5 to both sides, $(x^2 - 2x + 5) + 6(x^2 - 2x + 5)^{\frac{1}{2}} = 16$,

Completing, &c., $(x^2 - 2x + 5) + 6(x^2 - 2x + 5)^{\frac{1}{2}} + 9 = 25$;

$$\text{Whence } (x^2 - 2x + 5)^{\frac{1}{2}} = 2 \text{ or } -8,$$

Squaring both sides, $x^2 - 2x + 5 = 4$ or 64,

$$\text{And } x^2 - 2x = -1 \;\ldots\ldots\ldots\ldots\; (1)$$
$$\text{Also } x^2 - 2x = 59 \;\ldots\ldots\ldots\ldots\; (2)$$

Solving (1), we get $x = 1$, both values being equal.

Solving (2), „ $x = 1 \pm 2\sqrt{15}$.

(8) Given $x^2 - x + 5(2x^2 - 5x + 6)^{\frac{1}{2}} = \frac{1}{2}(3x + 33)$.

$$\text{Transposing, } x^2 - \frac{5}{2}x - \frac{33}{2} + 5(2x^2 - 5x + 6)^{\frac{1}{2}} = 0,$$

Multiplying by 2, and adding 39 to both sides, we have—

$$(2x^2 - 5x + 6) + 10(2x^2 - 5x + 6)^{\frac{1}{2}} = 39.$$

Completing and extracting the root—

$$\therefore (2x^2 - 5x + 6)^{\frac{1}{2}} + 5 = \pm 8,$$
$$(2x^2 - 5x + 6)^{\frac{1}{2}} = 3 \text{ or } -13,$$
$$\text{Squaring, } 2x^2 - 5x + 6 = 9 \text{ or } 169,$$
$$x^2 - \frac{5}{2}x = \frac{3}{2} \;\ldots\ldots\ldots\ldots\; (1)$$
$$x^2 - \frac{5}{2}x = \frac{163}{2} \;\ldots\ldots\ldots\ldots\; (2)$$

Solving (1), $x = 3$ or $-\frac{1}{2}$.

Solving (2), $x = \frac{5}{4} \pm \sqrt{\frac{1329}{16}} = \frac{1}{4}(5 \pm \sqrt{1329})$.

184. That a quadratic can have only two roots may be shown directly thus:

If possible, let m, n, p, be three different roots of the equation

$$x^2 + a x + b = 0.$$

$$\therefore m^2 + a m + b = 0. \quad \ldots\ldots\ldots \quad (1.)$$

$$n^2 + a n + b = 0. \quad \ldots\ldots\ldots\ldots \quad (2.)$$

$$p^2 + a p + b = 0. \quad \ldots\ldots\ldots\ldots \quad (3.)$$

Subtracting (2.) from (1.), we have,

$$(m^2 - n^2) + a (m - n) = 0,$$

or dividing every term by $(m - n)$,

$$m + n + a = 0.$$

Again subtracting (3.) from (1.), we have,

$$(m^2 - p^2) + a (m - p) = 0,$$

or dividing every term by $(m - p)$,

$$m + p + a = 0;$$

$$\therefore n = p, \text{ which is impossible.}$$

Therefore a quadratic cannot have more than two roots.

185. The two roots of every proper quadratic equation are equally real and significant, though the meaning of the negative quantity is not always apparent. In questions producing equations, the negative value of the answer is not generally even required according to the conditions proposed; but some modification of them is necessary to make it intelligible.

Ex. 1.—Several persons have to receive amongst them £800, but three of them being struck off the number before the money was paid, the rest receive £60 each additional. How many persons were concerned?

Let x = the number of persons—

Then $\frac{800}{x}$ = the share each had to receive at first,

$\frac{800}{x - 3}$ = „ „ actually received.

$$\therefore \frac{800}{x - 3} = \frac{800}{x} + 60 \text{ by the question.}$$

$$\text{When } x^2 - 3 x = 40 \text{ and } x = \frac{3}{2} \pm \frac{13}{2},$$

$$x = 8 \text{ and } - 5.$$

-5 is a solution of a modification of the question, which may be thus enunciated.

Several persons have to pay amongst them £800, but three being added to them, each of the original number has £60 less to pay. What number of persons were concerned?

Ex. 2.—A person *sold* a number of horses for £240, and if he had sold 4 *less* for the same money, the price would have been £2 *more* for each. How many did he *sell?*

Let x = the number of horses—

Then $\frac{240}{x}$ = the price at which he sold each,

And $\frac{240}{x-4}$ = the price at which he would have sold each; had he sold four less for the same money,

$$\therefore \frac{240}{x-4} = \frac{240}{x} + 2 \text{ by the question.}$$

$$\text{Whence } x^2 - 4x = 480,$$

$$x = 2 \pm \sqrt{484},$$

$$= 2 \pm 22 = 24 \text{ and } -20.$$

-20 is the answer of the following modification of the question.

A person *bought* a number of horses for £240, and if he had bought 4 *more* for the same money, the price would have been £2 *less* for each. How many did he *buy?*

EXAMPLES.

(1) Given $2x^2 + 10x + 24 = 124$ to find x.

$$\text{By transposition } 2x^2 + 10x = 100,$$

$$\text{Dividing by } 2,\ x^2 + 5x = 50,$$

$$\text{Completing the square } x^2 + 5x + \frac{25}{4} = 50 + \frac{25}{4} = \frac{225}{4},$$

$$\text{Extracting the root } x + \frac{5}{2} = \pm \frac{15}{2}.$$

$$\therefore x = 5 \text{ or } -10.$$

(2) Given $4x + \frac{2x-1}{x-2} = 3x + \frac{7x-5}{2}$ to find x.

Clearing the fractions,

$$8x^2 - 16x + 4x - 2 = 6x^2 - 12x + 7x^2 - 19x + 10,$$

$$\text{by transposition } -5x^2 + 19x = 12.$$

Changing the signs and dividing by 5,

$$x^2 - \frac{19}{5}x = -\frac{12}{5}.$$

$$\text{Completing the square, } x^2 - \frac{19}{5}x + \frac{361}{100} = -\frac{12}{5} + \frac{361}{100} = \frac{121}{100}.$$

Extracting the root, $x - \frac{19}{10} = \pm \frac{11}{10}$.

$$\therefore\ x = 3 \text{ or } \frac{4}{5}.$$

(3) Given $\sqrt{(2 + x)(10 - x)} = 3x - 18$, to find x.

Squaring both sides, $20 + 8x - x^2 = 9x^2 - 108x + 324$,

by transposition, $- 10x^2 + 116x = 304$.

$$\therefore\ x^2 - \frac{116}{10}x = - \frac{304}{10}.$$

Completing the square,

$$x^2 - \frac{116}{10}x + \left(\frac{58}{10}\right)^2 = - \frac{304}{10} + \left(\frac{58}{10}\right)^2 = \frac{324}{100}.$$

Extracting the root, $x - \frac{58}{10} = \pm \frac{18}{10}$.

$$\therefore\ x = \frac{76}{10} \text{ or } 4.$$

(4) Given $\sqrt{\frac{x}{x + 8}} + \sqrt{\frac{x + 8}{x}} = \frac{10}{3}$, to find x.

Squaring both sides, $\frac{x}{x + 8} + 2 + \frac{x + 8}{x} = \frac{100}{9}$.

Clearing the fractions and transposing, $- 64x^2 - 512x = - 576$,

Whence $x^2 + 8x = 9$.

Completing the square, $x^2 + 8x + 16 = 25$.

Extracting the root, $x + 4 = \pm 5$.

$$\therefore\ x = 1 \text{ or } - 9.$$

(5) Given $x + \frac{\sqrt{x - 3}}{2} = 8$, to find x.

Clearing the fractions, $2x + \sqrt{x - 3} = 16$.

Transposing, $\sqrt{x - 3} = 16 - 2x$.

Squaring both sides, $x - 3 = 256 - 64x + 4x^2$,

Whence $x^2 - \frac{65}{4}x = - \frac{259}{4}$.

Completing the square,

$$x^2 - \frac{65}{4}x + \left(\frac{65}{8}\right)^2 = - \frac{259}{4} + \left(\frac{65}{8}\right)^2 = \frac{81}{64}.$$

Taking the root, $x - \frac{65}{8} = \pm \frac{9}{8}$.

$$\therefore\ x = \frac{74}{8} \text{ or } 7.$$

(6) $$7\sqrt{2x^2-10x+3} = 72+5x-x^2.$$

$$2x^2-10x-144+14\sqrt{2x^2-10x+3} = 0.$$

$$(2x^2-10x+3)+14\sqrt{2x^2-10x+3} = 147,$$

a quadratic; solving we obtain,

$$\sqrt{2x^2-10x+3} = -7\pm\sqrt{49+147},$$
$$= 7 \text{ and } -21.$$

$$\sqrt{2x^2-10x+3} = 7$$
$$2x^2-10x+3 = 49$$
$$x^2-5x = 23$$
$$x = \frac{5}{2}\pm\sqrt{\frac{25}{4}+\frac{92}{4}}$$
$$= \frac{5}{2}\pm\sqrt{\frac{9\times 13}{4}}$$
$$= \frac{5\pm 3\sqrt{13}}{2}.$$

$$\sqrt{2x^2-10x+3} = -21$$
$$2x^2-10x+3 = 441$$
$$x^2-5x = 219$$
$$x = \frac{5}{2}\pm\sqrt{\frac{25}{4}+\frac{876}{4}}$$
$$= \frac{5}{2}\pm\sqrt{\frac{901}{4}}$$
$$= \frac{5\pm\sqrt{901}}{2}.$$

(7) $2x\sqrt{x^2+x-1} = 2x^2-5x+2$. (*B.A.* 1863.)

Squaring, $4x^2(x^2+x-1) = 4x^4-20x^3+33x^2-20x+4$.

Or $4x^4+4x^3-4x^2 = 4x^4-20x^3+33x^2-20x+4$.

Omit $4x^4$ on both sides, and transpose,

$$\therefore 24x^3-37x^2+20x-4 = 0.$$

$$\text{And } x^3-\frac{37}{24}x^2+\frac{5}{6}x-\frac{1}{6} = 0.$$

Separate this into its component factors,

$$\therefore \left(x-\frac{2}{3}\right)\left(x^2-\frac{7}{8}x+\frac{1}{4}\right) = 0.$$

From this we have,

$$x-\frac{2}{3} = 0 \;\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{And } x^2-\frac{7}{8}x+\frac{1}{4} = 0 \;\ldots\ldots\ldots\ldots (2)$$

From (1), $x = \frac{2}{3}$.

From (2), $x = \frac{7}{16}\pm\sqrt{\frac{49}{256}-\frac{1}{4}}$

$$= \frac{7}{16}\pm\frac{1}{16}\sqrt{-15}.$$

And the three values of x are,

$$x = \frac{2}{3}.$$
$$x = \frac{7+\sqrt{-15}}{16}.$$
$$x = \frac{7-\sqrt{-15}}{16}.$$

Second Method of Solution.

186. Assuming the general form as given above (Art. 176),

$$a\,x^2 + b\,x + c = 0.$$
$$\therefore\ a\,x^2 + b\,x = -\,c.$$

If, in order to avoid fractions, we multiply every term of the latter equation by four times the coefficient of the first term, and add to both sides the square of the coefficient of the second term, we shall have,

$$4\,a^2\,x^2 + 4\,a\,b\,x + b^2 = b^2 - 4\,a\,c,$$

of which the left side forms a complete square. Extracting the root of both sides,

$$2\,a\,x + b = \pm\sqrt{b^2 - 4\,a\,c}.$$
$$\therefore\ x = \frac{-\,b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}\ \ldots\ldots\ldots\ (A)$$

EXAMPLES.

(1) Given $3\,x^2 + 7\,x = 48$, to find x.

(× 12) $36\,x^2 + 84\,x = 576.$

Completing the square, $36\,x^2 + 84\,x + 7^2 = 625,$

(√) $6\,x + 7 = \pm\,25.$

$\therefore\ x = 3$ or $-\,5\tfrac{1}{3}.$

(2) Given $\sqrt{x + 3} = \dfrac{6}{\sqrt{x - 2}}$, to find x.

Squaring, $x + 3 = \dfrac{36}{x - 2},$

Whence $x^2 + x = 42.$

(× 4) $4\,x^2 + 4\,x = 168.$

Completing the square, $4\,x^2 + 4\,x + 1 = 169.$

(√) $2\,x + 1 = \pm\,13.$

$\therefore\ x = 6$ or $-\,7.$

(3) Given $\dfrac{5\,x - 4}{x + 2} + \dfrac{8\,x - 5}{x} = 10$, to find x.

Clearing the fractions and transposing, we get,

$3\,x^2 - 13\,x = 10.$

(× 12) $\therefore\ 36\,x^2 - 156\,x = 120.$

Completing the square, $36\,x - 156\,x + 13^2 = 289.$

(√) $6\,x - 13 = \pm\,17.$

$\therefore\ x = 5$ or $-\,\dfrac{2}{3}.$

187. Hence the most expeditious method of solving quadratic equations with one unknown quantity, is by substitution of the numerical coefficients of x in the preceding formula (A), as shown in the following examples.

Examples.

(1) Given $3x^2 - 19x + 28 = 0.$

$$(a = 3,\ -b = 19,\ c = 28.)$$

$$x = \frac{19 \pm \sqrt{361 - 336}}{6} = \frac{19 \pm 5}{6},$$

$$= 4 \text{ or } 2\tfrac{1}{3}.$$

(2) $$7x^2 + 13x - 102 = 0.$$

$$(a = 7,\ -b = -13,\ c = -102.)$$

$$x = \frac{-13 \pm \sqrt{169 + 2856}}{14} = \frac{-13 \pm 55}{14},$$

$$= 3 \text{ or } -4\tfrac{6}{7}.$$

(3) $$11x^2 - 13x - 210 = 0.$$

$$x = \frac{13 \pm \sqrt{169 + 9240}}{22} = \frac{13 \pm 97}{22},$$

$$= 5 \text{ or } -3\tfrac{9}{11}.$$

(4) $$-5x^2 + 19x - 12 = 0.$$

$$(a = -5,\ -b = -19,\ c = -12.)$$

$$x = \frac{-19 \pm \sqrt{361 - 240}}{-10} = \frac{-19 \pm 11}{-10},$$

$$= 3 \text{ or } \tfrac{4}{5}.$$

(5) $$-10x^2 + 116x - 304 = 0.$$

$\div (-2)$ $$5x^2 - 58x + 152 = 0.$$

$$x = \frac{58 \pm \sqrt{3364 - 3040}}{10} = \frac{58 \pm 18}{10},$$

$$= 7\tfrac{3}{5} \text{ or } 4.$$

(6) $$3x - \frac{169 - 3x}{x} = 29.$$ (*B. A.* 1859.)

$$3x^2 - 26x - 169 = 0.$$

$$x = \frac{26 \pm \sqrt{676 + 2028}}{6},$$

$$= \frac{26 \pm 52}{6} = 13 \text{ or } -4\tfrac{1}{3}.$$

(7) $$(x - a)(x - b) = (c - a)(c - b). \qquad (B.A.\ 1859.)$$

$$x^2 - (a + b)\,x + a\,b = c^2 - (a + b)\,c + a\,b.$$

$$x^2 - (a + b)\,x + (a + b - c)\,c = 0.$$

$$x = \frac{a + b \pm \sqrt{\overline{a + b}^2 - 4\,(a + b - c)\,c}}{2}$$

$$= \frac{a + b \pm (a + b - 2\,c)}{2} = a + b - c \text{ or } c.$$

(8) $$2\,x^2 + 11\,x + 15 = 0. \qquad (B.A.\ 1861.)$$

$$x = \frac{-11 \pm \sqrt{121 - 120}}{4} = -3 \text{ or } -2\tfrac{1}{2}.$$

(9) $$b\,x^2 + 3\,a\,x + b - a = a\,x^2 + 3\,b\,x + a - b. \qquad (B.A.\ 1862.)$$

$$\overline{a - b},\ x^2 - 3\,(a - b)\,x + 2\,(a - b) = 0.$$

$$\therefore\ x^2 - 3\,x + 2 = 0.$$

$$x = \frac{3 \pm \sqrt{9 - 8}}{2} = 2 \text{ or } 1.$$

(10) $$7\,x^2 + 5\,x - 38 = 0.$$

$$x = \frac{-5 \pm \sqrt{25 + 1064}}{14}$$

$$= \frac{-5 \pm 33}{14} = 2 \text{ or } -2\tfrac{5}{7}.$$

(11) $$13\,x^2 - 11\,x - 270 = 0.$$

$$x = \frac{11 \pm \sqrt{121 + 14040}}{26}$$

$$= \frac{11 \pm 119}{26} = 5 \text{ or } -4\tfrac{2}{13}.$$

(12) $$5\,x^2 + 43\,x + 24 = 0.$$

$$x = \frac{-43 \pm \sqrt{1849 - 480}}{10}$$

$$= \frac{-43 \pm 37}{10} = -8 \text{ or } -\tfrac{3}{5}.$$

Let the reader notice that in the last example, where there is no change of sign, both the roots are negative: in (1) (4) and (5), where the signs change at every term, both roots are positive: in (2) (3) (6) (10) and (11), where two consecutive terms have the same sign, one root is positive and the other negative; and, finally, when the two consecutive terms that have the same sign stand first, as in (2) and (7), the negative root is the larger; but when the two consecutive terms that have the same sign stand second and third, as in (3) (6) and (11), the positive root is the larger.

Signs of the terms.			Roots are—
x^2	x	Independent.	
+	+	+	both negative,
−	−	−	
−	+	−	both positive.
+	−	+	
+	+	−	Negative root numerically larger than the positive.
−	−	+	
+	−	−	Positive root numerically larger than the negative.
−	+	+	

QUADRATIC EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES.

188. When there are two equations and two unknown quantities, they cannot always be solved by the preceding methods, for their solution will, in some cases, depend on an equation of the fourth degree, containing the two unknown quantities. There are several kinds of such equations, however, the solution of which depends ultimately on that of a quadratic containing only one unknown quantity.

Examples.

(1) Given $x + y + xy = 11$ . . . . (1)

$x^2y + xy^2 = 30$ . . . . (2)

Eq. (2) is $(x + y)xy = 30$.

From (1) $x + y = 11 - xy$,

(2) $x + y = \frac{30}{xy}$.

Equating $11 - xy = \frac{30}{xy}$,

Whence $11xy - x^2y^2 = 30$,

And $x^2y^2 - 11xy = -30$.

$$\text{Solving } xy = \frac{11}{2} \pm \sqrt{\frac{121}{4} - \frac{120}{4}}$$

$$= \frac{11}{2} \pm \frac{1}{2}$$

$$= 6 \text{ and } 5.$$

Taking $xy = 6$, and substituting this value in either (1) or (2), we have,

$$x + y = 5,$$

and $x = 5 - y$, substituting in (1), we have,

$$5 - y + y + (5 - y)y = 11,$$

and $y^2 - 5y = -6$.

$$\text{Whence } y = \frac{5}{2} \pm \sqrt{\frac{25}{4} - \frac{24}{4}}$$

$$= \frac{5}{2} \pm \frac{1}{2} = 3 \text{ and } 2.$$

Again, taking $x\,y = 5$, the other value, and substituting in (1) or (2),

$$\therefore\ x + y = 6$$

$$x = 6 - y.$$

Substituting this value of x in (1),

$$6 - y + y + (6 - y)\,y = 11,$$

$$y^2 - 6\,y = -\,5,$$

$$y = 3 \pm \sqrt{9 - 5}$$

$$= 3 \pm 2 = 5 \text{ and } 1.$$

Therefore the four values of y are 3, 2, 5, 1,

And of x 2, 3, 1, 5.

(2) Given $x^2 + x\,y = a^2$ . . . . (1)

$y^2 - x\,y = b^2$ . . . . (2)

From (1) $x\,y = a^2 - x^2$,

(2) $x\,y = y^2 - b^2$.

Equating $a^2 - x^2 = y^2 - b^2$,

$$\therefore\ y^2 = a^2 - (x^2 - b^2),$$

$$y = \sqrt{a^2 - x^2 + b^2}.$$

Substituting this value of y in (1),

$$x^2 + \sqrt{a^2\,x^2 - x^4 + b^2\,x^2} = a^2,$$

$$\sqrt{a^2\,x^2 - x^4 + b^2\,x^2} = a^2 - x^2.$$

Squaring $a^2\,x^2 - x^4 + b^2\,x^2 = a^4 - 2\,a^2\,x^2 + x^4$,

$$\text{Whence } x^4 - \frac{3\,a^2 + b^2}{2}\,x^2 = -\,\frac{a^4}{2}.$$

$$\text{Solving } x^2 = \frac{3\,a^2 + b^2}{4} \pm \sqrt{\frac{9\,a^4 + 6\,a^2\,b^2 + b^4}{16} - \frac{8\,a^4}{16}},$$

$$x^2 = \frac{3\,a^2 + b^2}{4} \pm \frac{1}{4}\sqrt{a^4 + b^4 + 6\,a^2\,b^2},$$

$$x = \pm\left\{\frac{3\,a^2 + b^2}{4} \pm \frac{1}{4}\,\sqrt{a^4 + b^4 + 6\,a^2\,b^2}\right\}^{\frac{1}{2}},$$

$$= \pm\,\frac{1}{2}\,\{3\,a^2 + b^2 \pm \sqrt{a^4 + b^4 - 6\,a^2\,b^2}\}^{\frac{1}{2}}.$$

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

(1) Divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5. (*B.A.* 1857.)

Let x and $(60 - x)$ be the parts.

Then $x(60 - x) : x^2 + (60 - x)^2 :: 2 : 5$ by the question;

$$\text{whence } 5x(60 - x) = 2x^2 + 2(60 - x)^2.$$

Reducing, we get

$$x^2 - 60x = -800;$$
$$x^2 - 60x + (30)^2 = 900 - 800;$$
$$x - 30 = \pm\sqrt{900 - 800};$$
$$\therefore \quad x = 30 \pm \sqrt{900 - 800};$$
$$x = 30 \pm 10;$$
$$x = 40 \text{ and } 20.$$

Therefore the parts are 40 and 20.

(2) *A* and *B* distribute £60 each among a certain number of persons: *A* relieves 40 persons more than *B* does, and *B* gives to each 5 shillings more than *A*. How many persons did *A* and *B* respectively relieve?

Suppose *A* relieved $x + 40$,
B ditto x,

Then $\frac{1200}{x + 40}$ = the number of shillings *A* gave to each,

$\frac{1200}{x}$ = ditto *B* ditto,

$$\text{And } \frac{1200}{x} = \frac{1200}{x + 40} + 5 \text{ by the question.}$$

$$\text{Whence } x^2 + 40x = 9600,$$
$$x = 80 \text{ or } -120.$$

∴ *A* relieved 120, and *B* 80 persons.

(3) It is required to find four numbers in arithmetical progression, such that their common difference shall be 4, and their continued product 176985.

Let $x - 6$, $x - 2$, $x + 2$, $x + 6$, be the numbers.

$$\text{Then } (x - 6)(x + 6)(x - 2)(x + 2) = 176985 = a,$$
$$\text{or } (x^2 - 6^2)(x^2 - 2^2) = a.$$
$$\text{Whence } x^4 - 40x^2 = a - 144 = 176841,$$
$$x^2 = 20 \pm \sqrt{177241}$$
$$= 441 \text{ or } -401,$$
$$x = 21 \text{ or } \sqrt{-401}.$$

∴ 15, 19, 23, 27, are the numbers.

(4) A detachment from an army was marching in regular column, with 5 men more in depth than in front; but upon the enemy coming in sight, the front was increased by 845 men; and by this movement the detachment was drawn up in 5 lines. Required the number of men.

Let x = number of men in front,
∴ $x + 5$ = ditto in depth,
And $x(x + 5)$ = whole number of men.

But when the front is increased by 845, we will have,

$x + 845 =$ men in front,

$5 =$ men in depth by the question,

And $5\,(x + 845) =$ whole number of men.

Equating $x\,(x + 5) = 5\,(x + 845)$,

$$x^2 + 5\,x = 5\,x + 4225.$$

Whence $x = 65$,

And $4550 =$ the whole number of men.

(5) A vintner sold 7 doz. of sherry and 12 doz. of claret for £50. He sold of sherry 3 doz. more for £10 than he did of claret for £6. Required the price of each.

Let $x =$ the price in pounds of the sherry per doz.

$y =$ ditto claret ditto.

Then $7\,x + 12\,y = 50$ by the question (1).

Also $\frac{10}{x} =$ the number of dozens of sherry sold for £10,

$\frac{6}{y} =$ ditto claret ditto £6,

And $\frac{10}{x} = \frac{6}{y} + 3$ by the question.

Whence $10\,y = (3\,y + 6)\,x$.

But from (1) $x = \frac{50 - 12\,y}{7}$, substitute this in the line above,

$$\therefore\ 10\,y = (3\,y + 6)\left(\frac{50 - 12\,y}{7}\right).$$

Whence $y^2 - \frac{2}{9}\,y = \frac{25}{3}$,

$y = 3$ and $x = 2$.

(6) It is required to find three numbers in geometrical progression, such, that their sum shall be 7, and the sum of their squares 21.

Let $\frac{x}{y}$, x, and $x\,y$ be the numbers;

$$\text{Then } \frac{x}{y} + x + x\,y = 7 \ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{x^2}{y^2} + x^2 + x^2\,y^2 = 21 \ldots\ldots\ldots\ldots (2)$$

From (1) $\frac{x}{y} + x\,y = 7 - x$.

Squaring $\frac{x^2}{y^2} + 2\,x^2 + x^2\,y^2 = 49 - 14\,x + x^2$.

$$\frac{x^2}{y^2} + x^2 + x^2\,y^2 = 49 - 14\,x \ldots\ldots\ldots (3)$$

Equating (2) and (3), we have, $21 = 49 - 14\,x$,

$$x = 2.$$

Substituting for x in (1), we have, $\frac{2}{y} + 2 + 2\,y = 7$.

$$\text{Whence } y^2 - \frac{5}{2}\,y = -\,1.$$

$$y = \frac{5}{4} \pm \sqrt{\frac{25}{16} - 1}$$

$$= 2 \text{ or } \tfrac{1}{2}.$$

Taking $x = 2$ and $y = 2$, the numbers are 1, 2, 4.
Ditto $x = 2$ and $y = \frac{1}{2}$, ditto 4, 2, 1.

(7) The sum of 4 numbers in arithmetical progression is 20, and the sum of their squares 120: find them.

Let the numbers be $(x - 3\,y)$, $(x - y)$, $(x + y)$, $(x + 3\,y)$;
Then $(x - 3\,y) + (x - y) + (x + y) + (x + 3\,y) = 4\,x = 20$ by the question,

$$\therefore\ x = 5;$$

Also the sum of their squares $= 4\,x^2 + 20\,y^2 = 120$,

$$\therefore\ x^2 + 5\,y^2 = 30,$$

Substituting the value of x, we have $25 + 5\,y^2 = 30$.

$y = 1$. $\therefore$ 2, 4, 6, 8, are the numbers.

(8) A person bought a quantity of cloth of two sorts for £7 18*s*. For every yard of the better sort he gave as many shillings as he had yards in all: and for every yard of the worse as many shillings as there were yards of the better sort more than of the worse. And the whole price of the better sort was to the whole price of the worse, as 72 : 7. How many yards had he of each?

Let $x =$ the number of yards of the better,
$y =$ ditto worse.
Then $x + y =$ the price in shillings of a yard of the better,
$x - y =$ ditto worse.

Hence, by the question, we have,

$$x\,(x + y) + y\,(x - y) = 158 \,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\, (1)$$

$$x\,(x + y)\ :\ y\,(x - y) :: 72 : 7 \,.\,.\,.\,.\,.\,.\,.\, (2)$$

Putting $y = m\,x$ in (1) and (2), we have,

$$x\,(x + m\,x) + m\,x\,(x - m\,x) = 158,$$

$$x\,(x + m\,x)\ :\ m\,x\,(x - m\,x) :: 72 : 7.$$

Multiplying out, we have,

$$\text{For (1), } x^2 + m\,x^2 + m\,x^2 - m^2\,x^2 = 158 \,.\,.\,.\,.\,.\, (3)$$

$$(2),\ x^2 + m\,x^2\ :\ m\,x^2 - m^2\,x^2 :: 72 : 7,$$

$$\text{Or (2), } (1 + m)x^2\ :\ (m - m^2)\,x^2\ :: 72 : 7.$$

$$\text{Whence } \frac{(1+m)\,x^2}{(m-m^2)\,x^2}, \text{ or } \frac{1+m}{m-m^2} = \frac{72}{7},$$

$$\text{And } 72\,m - 72\,m^2 = 7 + 7\,m,$$

$$\text{From which } m^2 - \frac{65}{72}\,m = -\frac{7}{72},$$

$$m = \frac{65}{144} \pm \sqrt{\left(\frac{65}{144}\right)^2 - \frac{7}{72}}$$

$$= \frac{7}{9} \text{ and } \frac{1}{8};$$

Taking $m = \frac{7}{9}$, we have from (3),

$$\frac{158}{81}\,x^2 = 158.$$

Whence $x = 9$,

$\therefore\ y = 7.$

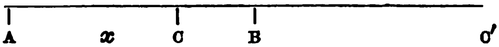
$$m = \frac{1}{8},$$

$$\frac{79}{64}\,x^2 = 158.$$

Whence $x = 8\sqrt{2}$,

$y = \sqrt{2}.$

(9) Two lights, *A* and *B*, situated at a distance from each other of 6 feet, are to each other in intensity as 4 to 1: required a point on the straight line joining them, which is equally illuminated by both.



Let us call the intensity of *B*, at the distance of one foot, unity or 1; then the intensity of *A* at the same distance will be 4; also, since the intensities of the same light at different distances are inversely as the squares of these distances, we have

$\frac{1}{1^2} : \frac{1}{x^2} :: 4 : \frac{4}{x^2}$, the intensity of *A* at the distance *x*, *i.e.*, at *C*.

$\frac{1}{1^2} : \frac{1}{(6-x)^2} :: 1 : \frac{1}{(6-x)^2}$, the intensity of *B* at the distance $(6 - x)$, *i.e.*, at *C*;

and these intensities are equal by the question;

$$\text{Or, } \frac{4}{x^2} = \frac{1}{(6-x)^2};$$

$$\text{Whence, } x = 12 \text{ or } 4.$$

The value 4 refers to the point *C* lying between *A* and *B*, and the value 12 to the point *C′* lying beyond *B*, and at either of these points the intensities are equal.

Verification.

The light A is 4, and $A\ C$ 4 feet. $\therefore$ intensity is as $\frac{4}{4^2} = \frac{1}{4}$,

" B is 1, and $B\ C$ 2 feet. $\therefore$ " is as $\frac{1}{2^2} = \frac{1}{4}$.

Again, A is 4, and $A\ C'$ 12 feet. $\therefore$ intensity is as $\frac{4}{12^2} = \frac{1}{36}$,

B is 1, and $B\ C'$ 6 feet. $\therefore$ " is as $\frac{1}{6^2} = \frac{1}{36}$.

(10) A person drops a stone into a well, and after 5 seconds hears it strike the water; find the depth to the surface of the water, supposing that a heavy body falls 16·1 feet in the first second of time; that the spaces through which the body falls are proportional to the squares of the times of falling; that the velocity of sound is 1127 feet in a second; and that the resistance of the air is neglected.

Let x = the depth required.

$$\text{Then } 16{\cdot}1 : x :: (1'')^2 : \frac{x}{16{\cdot}1}.$$

$\therefore \frac{x}{16{\cdot}1}$ is the square of the time of falling through x;

And $\sqrt{\frac{x}{16{\cdot}1}}$ is the time of (the stone's) falling, &c.

For brevity, let $1127 = n$.

$\therefore \frac{x}{n}$ is the time of the sound's passage.

And $\sqrt{\frac{x}{16{\cdot}1}} + \frac{x}{n} = 5''$ by the question;

$$\therefore \sqrt{\frac{x}{16{\cdot}1}} = 5 - \frac{x}{n}.$$

$$\text{Squaring } \frac{x}{16{\cdot}1} = 25 - \frac{10\,x}{n} + \frac{x^2}{n^2}.$$

$$\text{Whence } x^2 - \left(10\,n + \frac{n^2}{16{\cdot}1}\right)x = -\,25\,n^2.$$

$$x = 5\,n + \frac{n^2}{32{\cdot}2} \pm \sqrt{25\,n^2 + \frac{10\,n^3}{32{\cdot}2} + \left(\frac{n^2}{32{\cdot}2}\right)^2 - 25\,n^2}$$

$$= 5\,n + \frac{n^2}{32{\cdot}2} \pm \sqrt{\frac{10\,n^3}{32{\cdot}2} + \left(\frac{n^2}{32{\cdot}2}\right)^2}$$

$$= 5635 + 39445 \pm \sqrt{2000453175}$$

$$= 353{\cdot}58 \text{ feet nearly.}$$

The negative sign must be taken, because 5″, the whole time, is greater than the time of the sound's motion, which is

$$\frac{x}{n} \text{ or } 5 + \frac{n}{32{\cdot}2} \pm \sqrt{\frac{10\,n}{32{\cdot}2} + \frac{n^2}{(32{\cdot}2)^2}}.$$

But 5 is not greater than

$$5 + \frac{n}{32{\cdot}2} + \sqrt{\frac{10\,n}{32{\cdot}2} + \frac{n^2}{(32{\cdot}2)^2}} \text{ with the positive sign.}$$

Verification.

First, for the time in which the stone would reach the surface of the water, we have, since

$$\text{Space} = 16{\cdot}1\,(\text{time})^2$$

$$\therefore \text{time} = \sqrt{\frac{\text{space}}{16{\cdot}1}} = \sqrt{\frac{353{\cdot}58}{16{\cdot}1}} = 4''{\cdot}6863.$$

Again, for the time in which the sound would reach the top of the well, we have

$$\overset{\text{feet.}}{1127} : \overset{\text{feet}}{353{\cdot}58} :: 1'' : 0''{\cdot}3137$$

$$\text{And } 4''{\cdot}6863 + 0''{\cdot}3137 = 5''.$$

CHAPTER X.

THE NATURE AND USE OF LOGARITHMS.

LOGARITHMS.

189. If the powers of any number be written down in succession, and their respective indices affixed to them, there will be two progressions formed: the first, geometrical; and the second, arithmetical; as for instance, the successive powers of 2 and 3 are,

2^0, or 1^0, 2^1, 4^2, 8^3, 16^4, 32^5, 64^6, 128^7, 256^8, 512^9, 1024^{10}, &c.

3^0, or 1^0, 3^1, 9^2, 27^3, 81^4, 243^5, 729^6, 2187^7, 6561^8, 19683^9, &c.

In either of the above series, the product of any two numbers may be obtained by making use of the sum of their indices, and the number attached to the new index is the required product. The multiplication of any of the above numbers is thus reduced to the addition of their indices, and similarly the quotient of any two numbers is obtained by the subtraction of their indices. The powers or roots of any of these numbers are also obtained by the multiplication or division of their indices by the number of their required power or root.

In either series the exponents are the logarithms of the numbers to which they are attached, and the number, whose powers are thus made use of, is the base of these logarithms: thus, in the equation

$$64 = 2^6,$$

6 is the logarithm of 64, and 2 is the base. Or substituting the letters $N = b^p$, p is the log. N to the base b, and the logarithm of N to the base b is thus expressed, $\log._b N = p$.

The logarithm of a number, therefore, may be defined to be the index of the power to which the base must be raised, that the result may be equal to that number.

190. In Briggs's tables the base chosen is 10. In this system the log. $10 = 1$, log. $100 = 2$, log. $1000 = 3$; and the log. of $\cdot 1 = -1$, log. $\cdot 01 = -2$, log. $\cdot 001 = -3$, &c., and all the results that have been stated above respecting whole numbers, are true of negative logarithms.

191. The logarithms of 10, 100, ·1, ·01, &c., are integer numbers, and are called ***characteristics***, which are either ***positive*** or ***negative***, as the sign + or − is prefixed to them.

192. This constitutes one of the advantages of having the base of the system of logarithms coincident with the system of arithmetical notation. In Napier's system, and indeed whenever any other base is taken, every logarithm of a fraction is found by subtracting the logarithm of the denominator from that of the numerator.

In Briggs's system the logarithm of every decimal fraction is found in the table, and only requires the proper characteristic to be prefixed.

193. The characteristic is never inserted in modern tables of logarithms.

When positive, it is always one less than the number of digits in the number whose logarithm is sought, and

When negative, it is the number of places from the decimal point to the first significant figure.

Since the logarithm of a number between 1 and 10 must be less than 1, it will be a fractional quantity; and the logarithm of any number between 10 and 100, being between 1 and 2, will also be fractional. It is thus very obvious that the logarithms of most numbers will be fractional. The fractional part is called the *mantissa.*

194. Suppose $\cdot x$ to be the log. of 8, or $8 = 10^{\cdot x}$; then because $80 = 8 \times 10$, log. $80 =$ log. $8 +$ log. $10 = 1 + \cdot x$. Similarly, log. $800 = 2\ \cdot x$, log. $8000 = 3\ \cdot x$, log. $\cdot 8 = -1\ \cdot x$, log. $\cdot 08 = -2\ \cdot x$, &c.

From this it appears that any change in the decimal point attached to a number only changes the characteristic of its logarithm. An example will make this plain.

NUMBERS.	LOGARITHMS.
2651·	3·423410
265·1	2·423410
26·51	1·423410
2·651	0·423410
·2651	— 1·423410
·02651	— 2·423410
·002651	— 3·423410

NOTE.—The negative sign is usually written over the characteristic instead of before it; thus, $\bar{1}$·423410.

195. The following *resumé* of the properties of logarithms is important. (Refer to Art. 82.)

1°. Since $a = a^1$, where a may be the base of any system, it is plain that the logarithm of the base is always = 1.

2°. Also since $1 = a^0$, we have 0 = the logarithm of 1 in every system of logarithms.

3°. Again, since (Art. 63) $0 = \frac{1}{a^{\infty}} = a^{-\infty}$, it appears that the logarithm of 0 is an infinite negative quantity.

4°. The logarithm of the product of two numbers is the sum, and the logarithm of the quotient is the difference of the logarithms of the numbers.

For let $a^p = N$, $a^q = N'$, where N, N' are any two numbers, and p q their logarithms to the base a, then—

$$a^p \times a^q = a^{p+q} = N.N',$$

but by definition $p + q$ is the logarithm of $N.N'$ to the base a, or as it is usually written—

$$p + q = \log._a N.N';$$

$$\therefore \log._a N N' = \log._a N + \log\ N'.$$

In like manner—

$$\frac{N}{N'} = a^{p-q},$$

$$\text{and} \therefore \log._a \frac{N}{N'} = \log._a N - \log._a N'.$$

Hence the logarithm of a fraction is the difference between the logarithms of the numerator and denominator.

5°. The logarithm of $\frac{1}{N}$ and N, are the same with different signs. By what precedes—

$$\begin{aligned} \log._a \frac{1}{N} &= \log._a 1 - \log._a N, \\ &= 0 - \log._a N, \\ &= - \log._a N. \end{aligned}$$

6°. The logarithm of any power of a number, is the logarithm of the number multiplied by the index which expresses the power.

$$\begin{aligned} &\text{Suppose } a^p = N, \\ &\text{then } a^{vp} = N^v, \\ &\text{or } v\,p = \log._a N^v \text{ by definition;} \\ &\text{but } p = \log._a N, \\ &\therefore \log._a N^v = v \,.\, \log._a N. \end{aligned}$$

In like manner—

$$\log._a N^{\frac{1}{v}} = \frac{1}{v} \log._a N.$$

196. The principal use of logarithms is to facilitate the performance of the arithmetical operations of *Multiplication*, *Division*, *Involution*, and *Evolution*: more particularly in cases where the quantities employed consist of several figures, and near approximations to the true results are considered sufficient for the practical purposes to which they are applied.

197. *To find a rule for ascertaining the characteristic of the logarithm of any number.*

We have $\log._{10} 10^2 = 2$, and $\log._{10} 10^3 = 3$; hence the logarithm of any number between 100 and 1000 (*i.e.* of any number consisting of three digits), is between 2 and 3, and, therefore, the characteristic is 2; similarly for numbers of four digits, the characteristic is 3; and in the same way it appears that the characteristic is always one less than the number of digits.

If the number be decimal, we must have a negative characteristic, for—

$$\log_{10} 1 = 0.$$

$$\log_{10} \cdot 1 = \log_{10} \frac{1}{10} = \log_{10} \frac{1}{10^1} = \log_{10} 10^{-1} = -1.$$

$$\log_{10} \cdot 01 = \log_{10} \frac{1}{100} = \log_{10} \frac{1}{10^2} = \log_{10} 10^{-2} = -2.$$

Hence the logarithm of a number between 1 and ·1, is less than 0 and greater than -1, and may therefore be represented by $-1 +$ a mantissa: in like manner, the logarithm of a number between ·1 and ·01 will be $-2 +$ a mantissa; and, generally, the characteristic will be one greater than the number of ciphers which precede the first significant figure.

The above rule is sufficient whenever the mantissa is positive, which is here assumed to be the case.

⁂ As is explained in most tables of trigonometrical logarithms, the *tabular* logarithm is made greater than the true logarithm by 10; an arrangement adopted merely for convenience, to avoid the confusion of introducing negative signs.

In practice it is as easy to use the real, as the tabular logarithm, and by doing so the trouble of afterwards correcting, and the liability to error in doing it, are diminished.

The rule which we have adopted in the examples given of the solution of triangles, is as follows:—Always write down the *real*, never the *tabular* logarithm.

When looking for any angle corresponding to a logarithm, *add* 10 to it *mentally*, and when looking for the logarithm of a given angle, *subtract* 10 *mentally* from the number found in the table.

CHAPTER XI.

PLANE TRIGONOMETRY, AND THE MEASUREMENT OF HEIGHTS AND DISTANCES.

PLANE TRIGONOMETRY.

198. TRIGONOMETRY, as the word indicates, was originally restricted to the measurement of triangles; in its largest modern sense, however, it is much more extensively applied.

199. An angle in trigonometry is any opening made by two right lines which meet, and consequently a trigonometrical angle may be two right angles or greater than two right angles.

200. There are two independent methods of measuring angles. In trigonometry considered as a *science*, a unit of measurement is employed entirely distinct from the unit employed in trigonometry considered as an *art*.

201. The first, which is called the analytical or theoretical unit, is that angle of a circle which is subtended by an arc equal to its radius. This unit is to be remembered as an angle, and is the same for circles of all sizes.

202. The second, which in this country is invariably used in practice, is that angle which is subtended by the 360th part of the whole circumference. This angle is called a degree, marked 1°; its 60th part a minute, marked thus, $1'$; the 60th part of a minute a second, marked thus, $1''$. The angle subtended by the fourth part of the circumference, or the right angle, is therefore 90°; two right angles 180°, and so on.

203. This second unit, with its subdivisions, is also made use of for measuring the arc of a circle; and it is then a *linear* unit, and varies in length as the circle varies in size. Care should be taken to distinguish between the cases in which it is employed as an *angular* and those in which it is used as a *linear* unit; where it is employed to denote *the length of an arc*, and where to measure *the magnitude of an angle*.

This caution is principally necessary, in order to avoid a confusion of the two ideas, as no incorrect results would follow from an indiscriminate use of the terms *arc* and *angle*. All the formulæ that will be given respecting angular magnitude, will be found equally true for the corresponding functions of subtending arcs.

204. The English division of the circle into 360 degrees is called the *sexagesimal*, or (in reference to a right angle) the *nonagesimal* scale of division.

In the *French* or *centesimal* scale, the circle is divided into 100 degrees called grades, marked 1^g, each grade into 100 minutes, each minute into 100 seconds, marked as in the English scale.

205. The rule for changing the degrees, minutes, &c., of one scale, into corresponding degrees, minutes, &c., of the other, is thus obtained:

$$\text{First, } \frac{F}{E} = \frac{100}{90}, \text{ or } 1\frac{1}{9};$$

$$\therefore F = 1\frac{1}{9}E = E + \frac{1}{9}E \ldots\ldots\ldots \text{I.}$$

$$\text{Again, } \frac{E}{F} = \frac{90}{100}, \text{ or } \frac{9}{10};$$

$$\therefore E = \frac{9}{10}F = F - \frac{1}{10}F \ldots\ldots\ldots \text{II.}$$

I. To reduce English degrees, &c., to French grades, &c., reduce the given seconds and minutes to the decimal of a degree, prefix the degrees and add 1-9th of the whole.

Example.—How many degrees, minutes, &c., in the centesimal division of the quadrant correspond to 101° 2′ 34″ in the nonagesimal system? (*B.A.* 1843.)

$$\begin{array}{r}
60)34\cdot \\
\hline
60)2\cdot566 \\
\hline
101\cdot04278 = E \\
11\cdot22697 = \frac{E}{9} \\
\hline
112\cdot26975 \\
\hline
\end{array}$$

Ans. $112^g\ 26'\ 97\cdot5''$.

II. To reduce French grades, &c., to English degrees, &c., write down the given grades, minutes, &c., as a decimal number, and removing the decimal place one to the left, repeat the same in a second line. Subtract the lower from the upper line, and reduce the decimal part of the remainder.

Example.—Express $112^g\ 26'\ 97{\cdot}5''$ in the English scale.

$$\begin{array}{r} 112{\cdot}26975\ F \\ 11{\cdot}22697\ \frac{F}{10} \\ \hline 101{\cdot}04278 \\ 60 \\ \hline 2{\cdot}56680 \\ 60 \\ \hline 34{\cdot}008 \\ \hline \end{array}$$

Ans. $101^\circ\ 2'\ 34{\cdot}008''$.

206. In order to ascertain how many degrees of the sexagesimal scale, the theoretical unit, referred to in Art. 201, contains, we must assume two propositions proved elsewhere.

1st. The circumference and diameter of a circle are incommensurable quantities, but they have a constant ratio to each other, the same for all circles. This ratio has been calculated to many decimal places, the first five of which are all that are generally used, viz., 3·14159; this number is represented by the letter π: therefore, using r for *radius*, and C for *circumference*, we have:

$$\frac{C}{\text{diameter}},\ \text{or}\ \frac{C}{2\,r} = \pi;\ \therefore C = 2\,\pi\,.\,r.$$

2nd. It is proved in Euclid, book VI., prop. 33, that the angle of a circle varies as its arc, or in other words, that the *angle* is a measure of the *arc*, and *vice versâ*.

Now, let θ be the angle of a circle subtended by an arc equal to the radius; then

$$\begin{aligned} \theta : 360^\circ &:: r : 2\,\pi\,r, \\ \therefore \theta\,.\,2\,\pi\,r &= 360^\circ\,r, \\ \therefore \theta\,.\,2\,\pi &= 360^\circ, \\ \therefore \theta = \frac{360}{2\,\pi} &= 57{\cdot}2957795^\circ, \end{aligned}$$

which is the number of ordinary English degrees in what is called the analytical or theoretical unit of angular measurement.

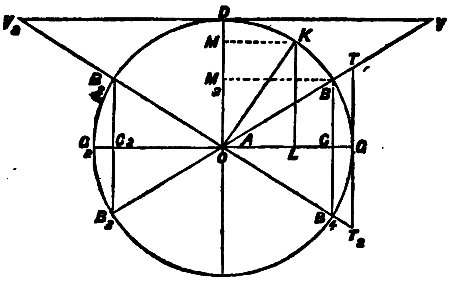
207. $\dfrac{arc}{radius}$, contracted into $\dfrac{a}{r}$, is the analytical measure of any angle, and very slight consideration will suffice to show that this fraction, multiplied into the number just found, gives the number of degrees in the sexagesimal or English scale, in any angle of a circle; thus—

$$\text{Any given angle} = \frac{a}{r}\,.\,57{\cdot}2957795^\circ.$$

208. The following six ratios, the alphabet of trigonometry, are *functions* of the angle to which they relate, *i. e.*, they vary as it varies.

First, let $B\ O\ C$ (Fig. I.) be any angle A; draw $B\ C$ perpendicular to $O\ C$, then

FIG. I.



$\frac{B\ C}{B\ O}$ is called the *sine* of A, which is written, *sin.* A.

$\frac{C\ O}{B\ O}$ . . *cosine* of A, . . *cos.* A.

$\frac{B\ C}{C\ O}$ . . *tangent* of A, . . *tan.* A.

$\frac{C\ O}{B\ C}$ . . *cotangent* of A, . . *cot.* A.

$\frac{B\ O}{C\ O}$ . . *secant* of A, . . *sec.* A.

$\frac{B\ O}{B\ C}$ . . *cosecant* of A, . . *cosec.* A.

NOTE.—To these are added, for the sake of completeness rather than for their utility, the *versed-sine*, and the *coversed-sine*; written *vers.* $A = 1 -$ *cos.* A, and *covers.* $A = 1 -$ *sin.* A.

209. We have before stated that the theoretical unit of angular measurement, in trigonometry, is that angle which has an arc equal to its radius. Now the line $B\ O$ is the radius, and therefore, the unit of measurement for the angle $B\ O\ C$; and if $B\ G$ be called the arc, then as $B\ G >, =,$ or $<, B\ O$, so is the angle $BO\ C$ greater than, equal to, or less than unity; and the proper measurement of the angle $B\ O\ C$, consequently, is $\frac{B\ G}{B\ O}$.

210. In old works on trigonometry, the radius was 1, and the line $B\ C$ itself represented the sine, the line $C\ O$ the cosine, and in the same way all the functions were represented by lines, as follows:—

sin. $A = B\ C.$
cos. $A = C\ O.$
tan. $A = T\ G.$
cot. $A = D\ V.$
sec. $A = O\ T.$
cosec. $A = O\ V.$
vers. $A = G\ C.$
covers. $A = D\ M^2.$

As by this method a new sine was required for even the same angle, for every change in the size of the circle, it has been abandoned, and the sine and other functions of an angle are no longer lines, but *ratios*, or *numbers*.

211. The expression $\frac{B\ C}{B\ O}$ represents the number of linear units in $B\ C$, divided by the number of linear units in $B\ O$, and the quotient of this fraction is the *number* which is called the sin. A.

Again, $\frac{B\ C}{B\ O}$, before reduction, whether it stands for the lines themselves, or the number of linear units they contain, is the *ratio* which is called the sin. A. The same remarks apply to all the other functions.

212. The derivation of the term sine, from *sinus*, the translation of an Arabic word, has reference to a bow-string which the double sine ($B B_4$, fig. I.) represents. Cosine is a contraction of "complementary sine," or "sine of the complement," where by complement is meant the excess of a right angle over the angle in question.

213. So cos. A = sin. of the complement of A,
cot. A = tan. ditto ditto
cosec. A = sec. ditto ditto
coversed sine A = versed sine do. ditto

214. To prove this, let the two triangles (Fig I.) $B\ O\ C$, $O\ K\ L$, be equal in all respects; namely, $K\ L$ to $C\ O$, $L\ O$ to $B\ C$, and $K\ O$ to $B\ O$; then, the angles $B\ O\ C$ and $K\ O\ L$ are equal to a right angle, and the angle $K\ O\ L$ is the complement of the angle $B\ O\ C$, and

cos. $K\ O\ L$ is $\frac{L\ O}{K\ O}$, and sin. $B\ O\ C$ is $\frac{B\ C}{B\ O}$;
cot. $K\ O\ L$ is $\frac{L\ O}{K\ L}$, and tan. $B\ O\ C$ is $\frac{B\ C}{C\ O}$;
cosec. $K\ O\ L$ is $\frac{K\ O}{K\ L}$, and sec. $B\ O\ C$ is $\frac{B\ O}{C\ O}$.

But, $\frac{L\ O}{K\ O}, \frac{L\ O}{K\ L}, \frac{K\ O}{K\ L}$, are respectively equal to $\frac{B\ C}{B\ O}, \frac{B\ C}{C\ O}, \frac{B\ O}{C\ O}$, and therefore, sin. $B\ O\ C$ = cos. $K\ O\ L$ = cos. of the complement of $B\ O\ C$. Similarly, tan. $B\ O\ C$, and sec. $B\ O\ C$, respectively equal the cot. and cosec. of the complement of $B\ O\ C$.

Generally, therefore, any function of an angle is also the co-function of its complement, and *vice versâ*.

215. The following results are necessary consequences of the previous definitions:

$$\left.\begin{array}{l}\text{Sin. } A \times \text{cosec. } A = \frac{B\,C}{B\,O} \times \frac{B\,O}{B\,C} = 1, \therefore \text{ sin. and cosec.} \\ \text{Cos. } A \times \text{sec. } A = \frac{C\,O}{B\,O} \times \frac{B\,O}{C\,O} = 1, \therefore \text{ cos and sec.} \\ \text{Tan. } A \times \text{cot. } A = \frac{B\,C}{C\,O} \times \frac{C\,O}{B\,C} = 1, \therefore \text{ tan. and cot.}\end{array}\right\}\text{are reciprocals.}$$

216. The signs + and − are used to point out opposition in direction. Hence all lines in a horizontal direction, measured from a given point O, to the right hand, are positive, and those to the left are negative; similarly, all perpendicular lines measured upwards are positive, downwards negative. The lines $B\ O$, $B_2\ O$, &c., which have no definite direction in space, are always positive.

217. If we now suppose the line $B\ O$ to revolve in the same plane round the centre O, and successively assume the positions, $B\ O$, $B_2\ O$, &c., and let fall the perpendiculars $B_2\ C_2$, $B_3\ C_2$, $B_4\ C$; then, when the angle is less than a right angle, as in $B\,O\,C$, all the six following functions are positive, because the lines $B\ C$ and $C\ O$ are positive, and $B\ O$, as has been stated, is positive.

$$\begin{array}{cccccc}\frac{+B\,C}{+B\,O}, & \frac{+C\,O}{+B\,O}, & \frac{+B\,C}{+C\,O}, & \frac{+C\,O}{+B\,C}, & \frac{+B\,O}{+C\,O}, & \frac{+B\,O}{+B\,C}. \\ +\text{ sin.} & +\text{ cos.} & +\text{ tan.} & +\text{ cot.} & +\text{ sec.} & +\text{ cosec.}\end{array}$$

When the angle is in the second right angle, or between one and two right angles, as in $B_2\ O\ C_2$, then the lines $B_2\ C_2$, $B_2\ O$, are positive, and $C_2\ O$ is negative, and the six functions have the following signs:

$$\begin{array}{cccccc}\frac{+B_2\,C_2}{+B_2\,O}, & \frac{-C_2\,O}{+B_2\,O}, & \frac{+B_2\,C_2}{-C_2\,O}, & \frac{-C_2\,O}{+B_2\,C_2}, & \frac{+B_2\,O}{-C_2\,O}, & \frac{+B_2\,O}{+B_2\,C_2}. \\ +\text{ sin.} & -\text{ cos.} & -\text{ tan.} & -\text{ cot.} & -\text{ sec.} & +\text{ cosec.}\end{array}$$

When the angle is in the third right angle, or between two and three right angles, $B_3\ C_2$ and $C_2\ O$ are negative, and $B_3\ O$ positive; and the six functions have the following signs:

$$\begin{array}{cccccc}\frac{-B_3\,C_2}{+B_3\,O}, & \frac{-C_2\,O}{+B_3\,O}, & \frac{-B_3\,C_2}{-C_2\,O}, & \frac{-C_2\,O}{-B_3\,C_2}, & \frac{+B_3\,O}{-C_2\,O}, & \frac{+B_3\,O}{-B_3\,C_2}. \\ -\text{ sin.} & -\text{ cos.} & +\text{ tan.} & +\text{ cot.} & -\text{ sec.} & -\text{ cosec.}\end{array}$$

When the angle is in the fourth right angle, or between three and four right angles, B_4C is negative, CO and B_4O are positive, and the six functions have the following signs:

$$\frac{-B_4C}{+B_4O}, \quad \frac{+CO}{+B_4O}, \quad \frac{-B_4C}{+CO}, \quad \frac{+CO}{-B_4C}, \quad \frac{+B_4O}{+CO}, \quad \frac{+B_4O}{-B_4C}$$
$$-\text{sin.} \quad +\text{cos.} \quad -\text{tan.} \quad -\text{cot.} \quad +\text{sec.} \quad -\text{cosec.}$$

218. These results should be committed to memory; and for this purpose the following tables have been formed, in which it is notable, that those functions which are reciprocals have the same signs throughout the four right angles.

Sin. and cosec. Tan. and cot. Sec. and cos.

$$\begin{array}{c|c} +_{2} & {}_{1}+ \\ \hline -^{3} & {}^{4}- \end{array} \qquad \begin{array}{c|c} -_{2} & {}_{1}+ \\ \hline +^{3} & {}^{4}- \end{array} \qquad \begin{array}{c|c} -_{2} & {}_{1}+ \\ \hline -^{3} & {}^{4}+ \end{array}$$

The numerals indicate the place of the first, second, &c., right angles.

219. The following will be readily understood:

Value of Angle.	Corresponding values of						
	sin.	cos.	tan.	cot.	sec.	cosec.	v. sin.
0°, symbolized thus, —	0	1	0	∞	1	∞	0
90°, or $\frac{\pi}{2}$, or ... ∟	1	0	∞	0	∞	1	1
180°, or π, or ... ⊥	0	− 1	0	−∞	− 1	−∞	2
270°, or $\frac{3\pi}{2}$, or ... ⌐	− 1	0	−∞	0	−∞	− 1	1

220. In art. 213, we proved that the sine of an angle was equal to the cosine of its complement, and *vice versâ*. We will now prove that the sine of an angle is equal to the sine of its supplement, where by supplement is meant, the excess of two right angles over the angle in question.

Observe in these cases that the angle A, whatever be its value, is always subtracted from 90° or 180°; thus the complement of 135° is − 45°, because 90° − 135° = − 45°; the supplement of 212° is − 32°, because 180° − 212° = − 32°.

Similarly the complement of $\left(A - \frac{\pi}{2}\right) = \pi - A$.

Ditto supplement of $\left(\frac{\pi}{2} + A\right) = \frac{\pi}{2} - A$.

Refer to Fig. I., and let the angle B_2OC_2 = angle BOC or A, and make $B_2O = BO$, and draw B_2C_2 perpendicular to C_2O.

Then $\angle C\,O\,B_2 = \pi - A$, *i.e.*, the supplement of A,

$$\text{and sin. } (\pi - A) = \frac{B_2\,C_2}{B_2\,O} = \frac{B\,C}{B\,O} = \text{sin. } A,$$

$$\text{also cos. } (\pi - A = \frac{C_2\,O}{B_2\,O} = \frac{-\,C\,O}{B\,O} = -\text{ cos. } A.$$

221. It therefore follows,

$$\text{Because } \frac{\pi}{2} - A \text{ is the supplement of } \frac{\pi}{2} + A,$$

$$\therefore \text{ sin. } \left(\frac{\pi}{2} + A\right) = \text{sin. } \left(\frac{\pi}{2} - A\right) = \text{cos. } A. \quad (218.)$$

$$\text{and cos. } \left(\frac{\pi}{2} + A\right) = -\text{ cos. } \left(\frac{\pi}{2} - A\right) = -\text{ sin. } A.$$

222. Again, because $-A$ is the supplement of $\pi + A$,

$$\therefore \text{ sin. } (\pi + A) = \text{sin. } (-A) = -\text{ sin. } A. \quad (220.)$$

$$\text{and cos. } (\pi + A) = -\text{ cos. } (-A) = -\text{ cos. } A.$$

223. If in Fig. I. we suppose the line $B\,O$ to coincide with the line $C\,O$, and thence to revolve round O as a centre, in the direction of $B\,O$, $B_2\,O$, and $B_3\,O$, till it arrive at $B_4\,O$, leaving the angle $B_4\,O\,C$ equal to the angle A, the angle thus described by the revolving line will be, $2\,\pi - A$; and hence,

$$\text{sin. } (2\,\pi - A) = \frac{B_4\,C}{B_4\,O} = \frac{-\,B\,C}{B\,O} = -\text{ sin. } A,$$

$$\text{cos. } (2\,\pi - A) = \frac{C\,O}{B_4\,O} = \frac{C\,O}{B\,O} = \text{cos. } A.$$

In conclusion, since after each complete revolution of the line $O\,B$ it again comes to its original position, and the lines $B\,C$, $C\,O$, are of the same magnitude and in the same direction, therefore the sines, cosines, and other functions undergo no change, either in sign or value, by the addition or subtraction of four right angles, or multiples of four right angles.

224. The following results of the preceding sections will be constantly assumed, with or without reference:

$$\text{The complement of } \left(A - \frac{\pi}{2}\right) = \pi - A.$$

$$\text{The supplement of } \left(\frac{\pi}{2} + A\right) = \frac{\pi}{2} - A.$$

$$\text{Sin. } (\pi - A) = \text{sin. } A., \text{ and cos. } (\pi - A) = -\text{ cos. } A.$$

$$\text{Sin. } \left(\frac{\pi}{2} + A\right) = \text{sin. } \left(\frac{\pi}{2} - A\right) = \text{cos. } A.$$

$$\text{Cos. } \left(\frac{\pi}{2} + A\right) = -\text{ cos. } \left(\frac{\pi}{2} - A\right) = -\text{ sin. } A.$$

$$\text{Sin. } (\pi + A) = \text{sin. } (-A) = -\text{ sin. } A.$$

$$\text{Cos. } (\pi + A) = -\cos. (-A) = -\cos. A.$$

$$\text{Sin. } (2\pi - A) = -\sin. A, \text{ and } \cos. (2\pi - A) = \cos. A.$$

225. (1) Refer to Fig. I., and by Euclid I., 47, we obtain the following results:

$$\left(\frac{C\,O}{B\,O}\right)^2 + \left(\frac{B\,C}{B\,O}\right)^2 = 1, \text{ or } \cos.^2 A + \sin.^2 A = 1;$$

$$\therefore \sin.^2 A = 1 - \cos.^2 A, \text{ and } \cos.^2 A = 1 - \sin.^2 A;$$

$$\therefore \sin. A = \sqrt{1 - \cos.^2 A}, \text{ and } \cos. A = \sqrt{1 - \sin.^2 A}.$$

(2) Tan. $A = \frac{B\,C}{C\,O}$, and dividing both terms of the fraction by $B\,O$, we obtain $\frac{B\,C}{B\,O}$ or sin. A for the numerator, and $\frac{C\,O}{B\,O}$ or cos. A for the denominator; therefore,

$$\text{Tan. } A = \frac{\sin. A}{\cos. A}.$$

(3) From $\left(\frac{C\,O}{B\,O}\right)^2 + \left(\frac{B\,C}{B\,O}\right)^2 = 1$, we have $1 + \left(\frac{B\,C}{C\,O}\right)^2 = \left(\frac{B\,O}{C\,O}\right)^2$ or $1 + \tan.^2 A = \sec.^2 A$. $\therefore \sec. A = \sqrt{1 + \tan.^2 A}$.

(4) Similarly, $1 + \left(\frac{C\,O}{B\,C}\right)^2 = \left(\frac{B\,O}{B\,C}\right)^2$, or $1 + \cot.^2 A = \text{cosec.}^2 A$;

$$\therefore \text{cosec. } A = \sqrt{1 + \cot.^2 A}.$$

(5) Similarly, cot. $A = \frac{\cos. A}{\sin. A}$, and dividing both terms of the fraction by cos. A, we obtain,

$$\text{Cot. } A = \frac{1}{\tan. A}.$$

226. Collecting these results, and such as are readily deducible from them, we have:—

$$\text{Sin. } A = \sqrt{1 - \cos.^2 A} = \frac{1}{\text{cosec. } A} \;\ldots\ldots\; (\alpha)$$

$$\text{Cos. } A = \sqrt{1 - \sin.^2 A} = \frac{1}{\sec. A} \;\ldots\ldots\; (\beta)$$

$$\text{Tan. } A = \frac{\sin. A}{\cos. A} = \frac{1}{\cot. A} \;\ldots\ldots\; (\gamma)$$

$$\text{Cot. } A = \frac{\cos. A}{\sin. A} = \frac{1}{\tan. A} \;\ldots\ldots\; (\delta)$$

$$\text{Sec. } A = \sqrt{1 + \tan.^2 A} = \frac{1}{\cos. A} \;\ldots\ldots\; (\epsilon)$$

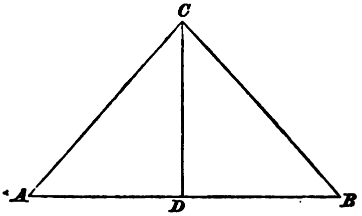
$$\text{Cosec. } A = \sqrt{1 + \cot.^2 A} = \frac{1}{\sin. A} \;\ldots\ldots\; (\zeta)$$

And in a similar manner may any of the trigonometrical functions of an angle be expressed in terms of any other function.

NUMERICAL VALUE OF TRIGONOMETRICAL FUNCTIONS.

227. We can find the numerical value of the sine of an angle of 30° by constructing an equilateral triangle *A B C*,

FIG. II.



and letting fall a perpendicular, *C D*, to the base, from *C;* then the angle *B A C*, by Euclid I., 32, corollary 1, is an angle of 60°, and the angle *A C D* an angle of 30°.

$$\therefore \text{ sin. } 30^\circ = \frac{A\,D}{A\,C} = \frac{\frac{1}{2}A\,B}{A\,C} = \frac{1}{2}.$$

$$\text{Again, cos. } 30^\circ = \sqrt{1 - \text{sin.}^2\, 30^\circ} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}.$$

$$\text{Also tan. } 30^\circ = \frac{\text{sin. } 30^\circ}{\text{cos. } 30^\circ} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

$$\text{Sec. } 30^\circ = \frac{1}{\text{cos. } 30^\circ} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}.$$

45°.

228. Since the sine of an angle is equal to the cosine of its complement, therefore

$$\text{Sin. } 45^\circ = \text{cos. } (90^\circ - 45^\circ) = \text{cos. } 45^\circ.$$

And as $\text{sin.}^2\, 45^\circ + \text{cos.}^2\, 45^\circ = 1$,

$$\therefore\ 2 \text{ sin.}^2\, 45^\circ = 1, \text{ and sin. } 45^\circ = \frac{1}{\sqrt{2}} = \text{cos. } 45^\circ.$$

$$\text{Tan. } 45^\circ = \frac{\text{sin. } 45^\circ}{\text{cos. } 45^\circ} = 1. \quad \text{Sec. } 45^\circ = \frac{1}{\text{cos. } 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

60°.

$$229. \quad \text{Sin. } 60^\circ = \text{cos. } (90^\circ - 60^\circ) = \text{cos. } 30^\circ = \frac{\sqrt{3}}{2}.$$

$$\text{Again, cos. } 60^\circ = \text{sin. } (90^\circ - 60^\circ) = \text{sin. } 30^\circ = \frac{1}{2}.$$

$$\text{Also, tan. } 60^\circ = \frac{\sin. 60^\circ}{\cos. 60^\circ} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}.$$

$$\text{Sec. } 60^\circ = \frac{1}{\cos. 60^\circ} = \frac{1}{\frac{1}{2}} = 2.$$

120°.

230. By Art. 220, sin. 120° = sin. (180° − 120°) = sin. 60°, and cos. 120° = − cos. (180° − 120°) = − cos. 60°;

$$\therefore \text{Sin. } 120^\circ = \frac{\sqrt{3}}{2}.$$

$$\text{Cos. } 120^\circ = -\frac{1}{2}.$$

$$\text{Tan. } 120^\circ = \frac{\sin. 120^\circ}{\cos. 120^\circ} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}.$$

$$\text{Sec. } 120^\circ = \frac{1}{\cos. 120^\circ} = \frac{1}{-\frac{1}{2}} = -2.$$

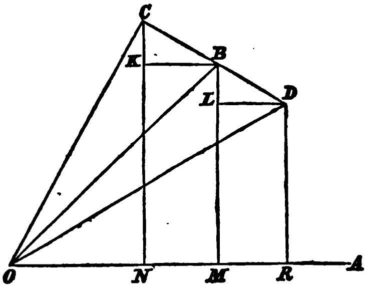
In a similar way, and by means of the formulæ to be hereafter given, the numerical values of all angles are found.

DEDUCTION OF FORMULÆ RESPECTING THE SUM AND DIFFERENCE OF TWO ANGLES.

231. Sin. $(A \pm B) = \sin. A \,.\, \cos. B \pm \cos. A \,.\, \sin. B$.

First let $\angle AOB = \angle A$, $\angle BOC = \angle B = \angle BOD$, then $\angle AOC = \angle (A + B)$, $\angle AOD = \angle (A - B)$.

FIG. III.



Draw CBD perpendicular to BO, and BM, CN, and DR, perpendicular to OA, also BK, DL, perpendicular to the parallels BM and CN; then the triangles BCK and DBL are equal in every respect, by Euclid, I., 26.

By Euclid, I., 29, the $\angle KBO = \angle AOB$;

and $\angle CBK = \angle CBO - \angle KBO = (\frac{\pi}{2} - A)$,

$\therefore$ sin. CBK = cos. A, and cos. CBK = sin. A.

(1) $$\text{Sin.}\,(A+B)=\frac{CN}{CO}=\frac{KN+CK}{CO}=\frac{BM}{CO}+\frac{CK}{CO},$$
$$=\frac{BM}{BO}\times\frac{BO}{CO}+\frac{CK}{BC}\times\frac{BC}{CO},$$
$$=\sin.\,A\,.\,\cos.\,B+\cos.\,A\,.\,\sin.\,B.$$

(2) $$\text{Sin.}\,(A-B)=\frac{DR}{DO}=\frac{BM-BL}{DO}=\frac{BM}{DO}-\frac{BL}{DO},$$
$$=\frac{BM}{BO}\times\frac{BO}{DO}-\frac{BL}{BD}\times\frac{BD}{DO},$$
$$=\sin.\,A\,.\,\cos.\,B-\cos.\,A\,.\,\sin.\,B.$$

$$\therefore\ \sin.\,(A\pm B)=\sin.\,A\,.\,\cos.\,B\pm\cos.\,A\,.\,\sin.\,B.$$

(3) $\text{Sin.}\,A=\sin.\,(\tfrac{1}{2}A+\tfrac{1}{2}A)$
$=\sin.\,\tfrac{1}{2}A\,.\,\cos.\,\tfrac{1}{2}A+\cos.\,\tfrac{1}{2}A\,.\,\sin.\,\tfrac{1}{2}A.$
$=2\sin.\,\tfrac{1}{2}A\,.\,\cos.\,\tfrac{1}{2}A.$
Similarly $\sin.\,2A=2\sin.\,A\,.\,\cos.\,A.$

232. $\text{Cos.}\,(A\pm B)=\cos.\,A\cos.\,B\mp\sin.\,A\sin.\,B.$

(1) $$\text{Cos.}\,(A+B)=\frac{NO}{CO}=\frac{MO-MN}{CO}=\frac{MO}{CO}-\frac{BK}{CO},$$
$$=\frac{MO}{BO}\times\frac{BO}{CO}-\frac{BK}{BC}\times\frac{BC}{CO},$$
$$=\cos.\,A\,.\,\cos.\,B-\sin.\,A\,.\,\sin.\,B.$$

(2) $$\text{Cos.}\,(A-B)=\frac{RO}{DO}=\frac{MO+MR}{DO}=\frac{MO}{DO}+\frac{DL}{DO},$$
$$=\frac{MO}{BO}\times\frac{BO}{DO}+\frac{DL}{DB}\times\frac{DB}{DO},$$
$$=\cos.\,A\,.\,\cos.\,B+\sin.\,A\,.\,\sin.\,B.$$

$$\therefore\ \cos.\,(A\pm B)=\cos.\,A\,.\,\cos.\,B\mp\sin.\,A\,.\,\sin.\,B.$$

(3) $\text{Cos.}\,A=\cos.\,(\tfrac{1}{2}A+\tfrac{1}{2}A)=\cos.\,\tfrac{1}{2}A\,.\,\cos.\,\tfrac{1}{2}A-\sin.\,\tfrac{1}{2}A\,.\,\sin.\,\tfrac{1}{2}A=\cos.^2\tfrac{1}{2}A-\sin.^2\tfrac{1}{2}A.$

$\text{Cos.}\,2A=\cos.\,(A+A)=\cos.^2A-\sin.^2A=(1-\sin.^2A)-\sin.^2A=1-2\sin.^2A.$

Also, $\cos.\,2A\,.=\cos.^2A-\sin.^2A=\cos.^2A-(1-\cos.^2A)=2\cos.^2A-1.$

233. $$\text{Tan.}\,(A\pm B)=\frac{\tan.\,A\pm\tan.\,B}{1\mp\tan.\,A\tan.\,B}.$$

(1) $$\text{Tan.}(A+B)=\frac{\sin.(A+B)}{\cos.(A+B)}=\frac{\sin.\,A\,.\cos.\,B+\cos.\,A\,.\sin.\,B}{\cos.\,A\,.\cos.\,B-\sin.\,A\,.\sin.\,B}.$$

Dividing both terms of the last fraction by $\cos.\,A\,.\,\cos.\,B$, we obtain,

$$\text{Tan.}\,(A+B)=\frac{\tan.\,A+\tan.\,B}{1-\tan.\,A\,.\,\tan.\,B}.$$

(2) $\text{Tan.}(A - B) = \frac{\sin.(A-B)}{\cos.(A-B)} = \frac{\sin. A . \cos. B - \cos. A . \sin. B}{\cos. A . \cos. B + \sin. A . \sin. B}.$

Dividing both terms of the last fraction by cos. A . cos. B, we obtain,

$$\text{Tan.}\ (A - B) = \frac{\tan. A - \tan. B}{1 + \tan. A \,.\, \tan. B}.$$

$$\therefore \text{Tan.}\ (A \pm B) = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A \,.\, \tan. B}.$$

(3) $\text{Tan.} 2\,A = \tan. (A + A) = \frac{\tan. A + \tan. A}{1 - \tan. A . \tan. A} = \frac{2 \tan. A}{1 - \tan.^2 A}$

(4) $\text{Sin.} 3 A = \sin. (2 A + A) = \sin. 2 A . \cos. A + \cos. 2 A . \sin. A.$

$= 2 \sin. A \cos. A . \cos. A + (1 - 2 \sin.^2 A) \sin. A.$

$= 2 \sin. A . \cos.^2 A + \sin. A - 2 \sin.^3 A.$

$= 2 \sin. A (1 - \sin.^2 A) + \sin. A - 2 \sin.^3 A.$

$= 2 \sin. A - 2 \sin.^3 A + \sin. A - 2 \sin.^3 A$

$= 3 \sin. A - 4 \sin.^3 A.$

(5) $\text{Cos.} 3 A. = \cos. (2 A + A) = \cos. 2 A . \cos. A - \sin. 2 A . \sin. A$

$= (2 \cos.^2 A - 1) \cos. A - 2 \sin. A . \cos. A . \sin. A$

$= 2 \cos.^3 A - \cos. A - 2 \cos. A . \sin.^2 A$

$= 2 \cos.^3 A - \cos. A - 2 \cos. A (1 - \cos.^2 A)$

$= 2 \cos.^3 A - \cos. A - 2 \cos. A + 2 \cos.^3 A$

$= 4 \cos.^3 A - 3 \cos. A.$

ADDITIONAL EXAMPLES.

(1) Show that $1 - 2 \sin.^2 (45^\circ - \frac{1}{2} A) = \sin. A.$

$\text{Sin.} (45^\circ - \frac{1}{2} A) = \sin. 45^\circ . \cos. \frac{1}{2} A - \cos. 45^\circ . \sin. \frac{1}{2} A$

$= \sin. 45^\circ (\cos. \frac{1}{2} A - \sin. \frac{1}{2} A)$

$= \frac{1}{\sqrt{2}} (\cos. \frac{1}{2} A - \sin. \frac{1}{2} A).$ Square both sides.

$\therefore \text{Sin.}^2 (45^\circ - \frac{1}{2} A) = \frac{1}{2} (\cos.^2 \frac{1}{2} A - 2 \sin. \frac{1}{2} A . \cos. \frac{1}{2} A + \sin.^2 \frac{1}{2} A)$

$= \frac{1}{2} (1 - 2 \sin. \frac{1}{2} A . \cos. \frac{1}{2} A) \quad \because \cos.^2 \frac{1}{2} A + \sin.^2 \frac{1}{2} A = 1.$

$2 \sin.^2 (45^\circ - \frac{1}{2} A) = 1 - 2 \sin. \frac{1}{2} A . \cos. \frac{1}{2} A.$ Transpose, &c.

$\therefore 1 - 2 \sin.^2 (45 - \frac{1}{2} A) = 2 \sin. \frac{1}{2} A . \cos. \frac{1}{2} A = \sin. A.$

(2) Show that $\frac{\tan. (45^\circ + \frac{1}{2} A) - \tan. (45^\circ - \frac{1}{2} A)}{\tan. (45^\circ + \frac{1}{2} A) + \tan. (45^\circ - \frac{1}{2} A)} = \sin. A.$

$\tan. (45^\circ + \frac{1}{2} A) = \frac{\tan. 45^\circ + \tan. \frac{1}{2} A}{1 - \tan. 45^\circ \tan. \frac{1}{2} A} = \frac{1 + \tan. \frac{1}{2} A}{1 - \tan. \frac{1}{2} A};$

$\tan. (45^\circ - \frac{1}{2} A) = \frac{\tan. 45^\circ - \tan. \frac{1}{2} A}{1 + \tan. 45^\circ \tan. \frac{1}{2} A} = \frac{1 - \tan. \frac{1}{2} A}{1 + \tan. \frac{1}{2} A}.$

Whence $\tan. (45^\circ + \frac{1}{2} A) - \tan. (45 - \frac{1}{2} A) = \frac{1 + \tan. \frac{1}{2} A}{1 - \tan. \frac{1}{2} A} - \frac{1 - \tan. \frac{1}{2} A}{1 + \tan. \frac{1}{2} A}$

$= \frac{(1 + \tan. \frac{1}{2} A)^2 - (1 - \tan. \frac{1}{2} A)^2}{1 - \tan.^2 \frac{1}{2} A} = \frac{4 \tan. \frac{1}{2} A}{1 - \tan.^2 \frac{1}{2} A}$

= the numerator of the given fraction.

Also $\tan.(45^\circ+\frac{1}{2}A)+\tan.(45^\circ-\frac{1}{2}A)=\frac{1+\tan.\frac{1}{2}A}{1-\tan.\frac{1}{2}A}+\frac{1-\tan.\frac{1}{2}A}{1+\tan.\frac{1}{2}A}$

$$=\frac{(1+\tan.\frac{1}{2}A)^2+(1-\tan.\frac{1}{2}A)^2}{1-\tan.^2\frac{1}{2}A}=\frac{2\,(1+\tan.^2\frac{1}{2}A).}{1-\tan.^2\frac{1}{2}A}$$

= the denominator of the given fraction.

$$\therefore\ \frac{\tan.(45^\circ+\frac{1}{2}A)-\tan.(45^\circ-\frac{1}{2}A)}{\tan.(45^\circ+\frac{1}{2}A)+\tan.(45^\circ-\frac{1}{2}A)}=\frac{4\tan.\frac{1}{2}A}{1-\tan.^2\frac{1}{2}A}\div\frac{2(1+\tan.^2\frac{1}{2}A)}{1-\tan.^2\frac{1}{2}A}$$

$=\frac{2\tan.\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}$. Divide every term by tan. $\frac{1}{2}A$, and it becomes,

$$=\frac{2}{\cot.\frac{1}{2}A+\tan.\frac{1}{2}A}=\frac{2}{\frac{\cos.\frac{1}{2}A}{\sin.\frac{1}{2}A}+\frac{\sin.\frac{1}{2}A}{\cos.\frac{1}{2}A}}=\frac{2}{\frac{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}{\sin.\frac{1}{2}A.\cos.\frac{1}{2}A}}$$

$$=2\sin.\tfrac{1}{2}A\,.\cos.\tfrac{1}{2}A=\sin.A.$$

(3) Show that $\frac{1-\tan.^2\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}=\cos.A.$

$$\frac{1-\tan.^2\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}=\frac{1-\frac{\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}}{1+\frac{\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}}=\frac{\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}}{\frac{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A}}$$

$$=\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}=\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A\,.}{1}$$

$$=\cos.\tfrac{1}{2}A\,.\cos.\tfrac{1}{2}A-\sin.\tfrac{1}{2}A\,.\sin.\tfrac{1}{2}A=\cos.(\tfrac{1}{2}A+\tfrac{1}{2}A)=\cos.A\,.$$

(4) Prove $\frac{\cot.\frac{1}{2}A-\tan\frac{1}{2}A}{\cot.\frac{1}{2}A+\tan.\frac{1}{2}A}=\cos.A\,.$

$$\frac{\cot.\frac{1}{2}A-\tan.\frac{1}{2}A}{\cot.\frac{1}{2}A+\tan.\frac{1}{2}A}=\frac{\frac{\cos.\frac{1}{2}A}{\sin.\frac{1}{2}A}-\frac{\sin.\frac{1}{2}A}{\cos.\frac{1}{2}A}}{\frac{\cos.\frac{1}{2}A}{\sin.\frac{1}{2}A}+\frac{\sin.\frac{1}{2}A}{\cos.\frac{1}{2}A}}=\frac{\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\sin.\frac{1}{2}A.\cos.\frac{1}{2}A}}{\frac{\cos.^2\frac{1}{2}A+\sin.^2\frac{1}{2}A}{\sin.\frac{1}{2}A\,.\cos.\frac{1}{2}A}}$$

$$=\frac{\cos.^2\frac{1}{2}A-\sin.^2\frac{1}{2}A}{\sin.\frac{1}{2}A\,.\cos.\frac{1}{2}A}\div\frac{1}{\sin.\frac{1}{2}A.\cos.\frac{1}{2}A}=\cos.^2\tfrac{1}{2}A-\sin.^2\tfrac{1}{2}A$$

= cos. A, as in the question preceding.

(5) Prove $\frac{1}{1+\tan.A\,.\tan.\frac{1}{2}A}=\cos.A.$

We first get tan. A in terms of tan $\frac{1}{2}A$, as follows,

$$\tan.A=\tan.(\tfrac{1}{2}A+\tfrac{1}{2}A)=\frac{\tan.\frac{1}{2}A+\tan.\frac{1}{2}A}{1-\tan.\frac{1}{2}A\,.\tan.\frac{1}{2}A}=\frac{2\tan.\frac{1}{2}A}{1-\tan.^2\frac{1}{2}A}$$

Substituting for this, we have

$$\frac{1}{1+\frac{2\tan.\frac{1}{2}A}{1-\tan.^2\frac{1}{2}A}\times\tan.\frac{1}{2}A}=\frac{1}{1+\frac{2\tan.^2\frac{1}{2}A}{1-\tan.^2\frac{1}{2}A}}=\frac{1}{\frac{1+\tan.^2\frac{1}{2}A}{1-\tan.^2\frac{1}{2}A}}$$

$=\frac{1-\tan.^2\frac{1}{2}A}{1+\tan.^2\frac{1}{2}A}$. Same as Example 3.

(6) Prove $\dfrac{2}{\text{tan.}\,(45^\circ + \frac{1}{2}A) + \text{cot.}\,(45^\circ + \frac{1}{2}A)} = \text{cos.}\,A$.

For $\text{tan.}\,(45^\circ + \frac{1}{2}A) = \dfrac{1 + \text{tan.}\,\frac{1}{2}A}{1 - \text{tan.}\,\frac{1}{2}A}$. See Question 2.

And $\text{cot.}\,(45^\circ + \frac{1}{2}A) = \dfrac{1}{\text{tan.}\,(45^\circ + \frac{1}{2}A)}$ (226 β).

$$\therefore \frac{2}{\text{tan.}\,(45^\circ + \frac{1}{2}A) + \text{cot.}\,(45^\circ + \frac{1}{2}A)} = \frac{2}{\dfrac{1+\text{tan.}\frac{1}{2}A}{1-\text{tan.}\frac{1}{2}A} + \dfrac{1-\text{tan.}\frac{1}{2}A}{1+\text{tan.}\frac{1}{2}A}}$$

$$= \frac{2}{\dfrac{(1 + \text{tan.}\,\frac{1}{2}A)^2 + (1 - \text{tan.}\,\frac{1}{2}A)^2}{(1 - \text{tan.}\,\frac{1}{2}A)\,(1 + \text{tan.}\,\frac{1}{2}A)}} = \frac{2}{\dfrac{2\,(1 + \text{tan.}^2\,\frac{1}{2}A)}{1 - \text{tan.}^2\,\frac{1}{2}A}}$$

$$= \frac{2\,(1 - \text{tan.}^2\,\frac{1}{2}A)}{2\,(1 + \text{tan.}^2\,\frac{1}{2}A)} = \frac{1 - \text{tan.}^2\,\frac{1}{2}A}{1 + \text{tan.}^2\,\frac{1}{2}A}.$$ Then as Example 3.

(7) Prove $2\,\text{cos.}\,(45^\circ + \frac{1}{2}A)\,\text{cos.}\,(45^\circ - \frac{1}{2}A) = \text{cos.}\,A$.

Expanding, we have

$$2\,\text{cos.}(45^\circ + \tfrac{1}{2}A)\,\text{cos.}(45^\circ - \tfrac{1}{2}A) = 2\,(\text{cos.}\,45^\circ\,\text{cos.}\,\tfrac{1}{2}A - \text{sin.}\,45^\circ\,\text{sin.}\,\tfrac{1}{2}A)$$
$$(\text{cos.}\,45^\circ\,\text{cos.}\,\tfrac{1}{2}A + \text{sin.}\,45^\circ\,\text{sin.}\,\tfrac{1}{2}A);$$

the right side of this equation is of the form,

$2\,(a - b)\,(a + b)$, which is equal to $2\,(a^2 - b^2)$.

Applying this principle to it, it becomes

$$2\,(\text{cos.}^2\,45^\circ\,\text{cos.}^2\,\tfrac{1}{2}A - \text{sin.}^2\,45^\circ\,\text{sin.}^2\,\tfrac{1}{2}A)$$
$$= 2\,\text{cos.}^2\,45^\circ\,(\text{cos.}^2\,\tfrac{1}{2}A - \text{sin.}^2\,\tfrac{1}{2}A) = \text{cos.}^2\,\tfrac{1}{2}A - \text{sin.}^2\,\tfrac{1}{2}A.$$

(since $2\,\text{cos.}^2\,45^\circ = 1$). Then as at the end of Ex. 3, or 232 (3).

(8) Prove $\dfrac{2}{\text{cot.}\,\frac{1}{2}A - \text{tan.}\,\frac{1}{2}A} = \text{tan.}\,A$.

$$\frac{2}{\text{cot.}\,\frac{1}{2}A - \text{tan.}\,\frac{1}{2}A} = \frac{2}{\dfrac{\text{cos.}\,\frac{1}{2}A}{\text{sin.}\,\frac{1}{2}A} - \dfrac{\text{sin.}\,\frac{1}{2}A}{\text{cos.}\,\frac{1}{2}A}} = \frac{2}{\dfrac{\text{cos.}^2\,\frac{1}{2}A - \text{sin.}^2\,\frac{1}{2}A}{\text{sin.}\,\frac{1}{2}A\,.\,\text{cos.}\,\frac{1}{2}A}}$$

$$= \frac{2\,\text{sin.}\,\frac{1}{2}A\,.\,\text{cos.}\,\frac{1}{2}A}{\text{cos.}^2\,\frac{1}{2}A - \text{sin.}^2\,\frac{1}{2}A} = \frac{\text{sin.}\,A}{\text{cos.}\,A} = \text{tan.}\,A.$$

(9) Prove $\text{cot.}\,A - 2\,\text{cot.}\,2A = \text{tan.}\,A$.

First, $\text{cot.}\,2A = \dfrac{1}{\text{tan.}\,2A} = \dfrac{1 - \text{tan.}^2 A}{2\,\text{tan.}\,A.}$ (233 (3)). Substituting,

we get $\text{cot.}\,A - 2\,\text{cot.}\,2A = \dfrac{1}{\text{tan.}\,A} - \dfrac{2\,(1 - \text{tan.}^2 A)}{2\,\text{tan.}\,A}$

$$= \frac{1}{\text{tan.}\,A} - \frac{1 - \text{tan.}^2 A}{\text{tan.}\,A} = \frac{1 - (1 - \text{tan.}^2 A)}{\text{tan.}\,A} = \text{tan.}\,A.$$

(10) Prove $\frac{1}{2}\,\text{tan.}\,(45^\circ + \frac{1}{2}A) - \frac{1}{2}\,\text{tan.}\,(45^\circ - \frac{1}{2}A) = \text{tan}\,A$.

$$\tfrac{1}{2}\,\text{tan.}(45^\circ + \tfrac{1}{2}A) - \tfrac{1}{2}\,\text{tan.}(45^\circ - \tfrac{1}{2}A) = \frac{1 + \text{tan.}\frac{1}{2}A}{2(1 - \text{tan.}\frac{1}{2}A)} - \frac{1 - \text{tan.}\frac{1}{2}A}{2(1 + \text{tan.}\frac{1}{2}A)}$$

$$= \frac{2\,(1 + \text{tan.}\,\frac{1}{2}A)^2 - 2\,(1 - \text{tan.}\,\frac{1}{2})^2}{4\,(1 - \text{tan.}^2\,\frac{1}{2}A)}.$$ Square and cancel.

$$\therefore = \frac{8\,\text{tan.}\,\frac{1}{2}A}{4\,(1 - \text{tan.}^2\,\frac{1}{2}A)} = \frac{2\,\text{tan.}\,\frac{1}{2}A}{1 - \text{tan.}^2\,\frac{1}{2}A} = \text{tan.}\,(\tfrac{1}{2}A + \tfrac{1}{2}A) = \text{tan.}\,A.$$

(11) Prove $\cot.^2 A \,.\, \cos.^2 A = \cot.^2 A - \cos.^2 A$.

$$\cot.^2 A \,.\, \cos.^2 A = \cot.^2 A\,(1 - \sin.^2 A) = \cot.^2 A - \cot.^2 A \,.\, \sin.^2 A$$

$$= \cot.^2 A - \frac{\cos.^2 A}{\sin.^2 A} \times \sin.^2 A = \cot.^2 A - \cos.^2 A.$$

(12) Prove $\sec.^2 A \,.\, \operatorname{cosec.}^2 A = \sec.^2 A + \operatorname{cosec.}^2 A$.

$$\sec.^2 A \,.\, \operatorname{cosec.}^2 A = (1 + \tan.^2 A)\operatorname{cosec.}^2 A \quad (226\ \delta).$$

$$= \operatorname{cosec.}^2 A + \operatorname{cosec.}^2 A \,.\, \tan.^2 A = \operatorname{cosec.}^2 A \,. + \frac{1}{\sin.^2 A} \times \frac{\sin.^2 A}{\cos.^2 A}$$

$$= \operatorname{cosec.}^2 A + \frac{1}{\cos.^2 A} = \operatorname{cosec.}^2 A + \sec.^2 A.$$

(13) Prove $\cos. A = \cos.^4 \tfrac{1}{2} A - \sin.^4 \tfrac{1}{2} A$.

$$\cos. A = \cos.^2 \tfrac{1}{2} A - \sin.^2 \tfrac{1}{2} A.$$

But $1 = \cos.^2 \tfrac{1}{2} A + \sin.^2 \tfrac{1}{2} A$. Multiplying these equations,

$$1 \times \cos. A = (\cos.^2 \tfrac{1}{2} A - \sin.^2 \tfrac{1}{2} A)(\cos.^2 \tfrac{1}{2} A + \sin.^2 \tfrac{1}{2} A)$$

$$= \cos.^4 \tfrac{1}{2} A - \sin.^4 \tfrac{1}{2} A.$$

(14) Prove $\dfrac{\tan. A + \tan. B}{\cot. A + \cot. B} = \tan. A \,.\, \tan. B$.

$$\frac{\tan. A + \tan. B}{\cot. A + \cot. B} = \frac{\dfrac{\sin. A}{\cos. A} + \dfrac{\sin. B}{\cos. B}}{\dfrac{\cos. A}{\sin. A} + \dfrac{\cos. B}{\sin. B}} = \frac{\dfrac{\sin. A \,.\, \cos. B + \cos. A \,.\, \sin. B}{\cos. A \,.\, \cos. B \,.}}{\dfrac{\sin. A \,.\, \cos. B + \cos. A \,.\, \sin. B}{\sin. A \,.\, \sin. B}}$$

$$= \frac{\sin. (A + B)}{\cos. A \,.\, \cos. B} \div \frac{\sin. (A + B)}{\sin. A \,.\, \sin. B} = \frac{\sin. A \,.\, \sin. B}{\cos. A \,.\, \cos. B}$$

$$= \tan. A \,.\, \tan. B.$$

(15) Prove $\tan. 2\,A = \dfrac{\sin. A + \sin. 3\,A}{\cos. A + \cos. 3\,A}$.

$$\tan. 2\,A = \frac{\sin. 2\,A}{\cos. 2\,A} = \frac{2 \sin. A \,.\, \cos. A}{\cos.^2 A - \sin.^2 A}.$$

Multiplying every term by $2 \cos. A$,

$$= \frac{4 \sin. A \,.\, \cos.^2 A}{2 \cos.^3 A - 2 \cos. A \sin.^2 A} = \frac{4 \sin. A\,(1 - \sin.^2 A)}{2 \cos.^3 A - 2 \cos. A\,(1 - \cos.^2 A)}$$

$$= \frac{4 \sin. A - 4 \sin.^3 A}{2 \cos.^3 A - 2 \cos. A + 2 \cos.^3 A} = \frac{4 \sin. A - 4 \sin.^3 A}{4 \cos.^3 A - 2 \cos. A}$$

$$= \frac{\sin. A + 3 \sin. A - 4 \sin.^3 A}{\cos. A + 4 \cos.^3 A - 3 \cos. A} = \frac{\sin. A + \sin. 3\,A}{\cos. A + \cos. 3\,A}.$$

(16) Prove $\sec. 2\,A = \dfrac{\cot. A + \tan. A}{\cot. A - \tan. A}$.

$$\text{Sec. } 2\,A = \frac{1}{\cos. 2\,A} = \frac{1}{\cos.^2 A - \sin.^2 A} = \frac{\cos.^2 A + \sin.^2 A}{\cos.^2 A - \sin.^2 A}.$$

Divide every term by $\cos. A \,.\, \sin. A$, and it becomes

$$\frac{\dfrac{\cos.^2 A}{\cos. A \,.\, \sin. A} + \dfrac{\sin.^2 A}{\cos. A \,.\, \sin. A}}{\dfrac{\cos.^2 A}{\cos. A \,.\, \sin. A} - \dfrac{\sin.^2 A}{\cos. A \,.\, \sin. A}} = \frac{\dfrac{\cos. A}{\sin. A} + \dfrac{\sin. A}{\cos. A}}{\dfrac{\cos. A}{\sin. A} - \dfrac{\sin. A}{\cos. A}} = \frac{\cot. A + \tan. A}{\cot. A - \tan. A}.$$

(17) Prove $\tan. 3A = \dfrac{\sin. A + \sin. 3A + \sin. 5A}{\cos. A + \cos. 3A + \cos. 5A}$.

We must first find sin. 5 A and cos. 5 A.

$\sin. (A + B) = \sin. A . \cos. B + \cos. A . \sin. B$

$\sin. (A - B) = \sin. A . \cos. B - \cos. A . \sin. B$, adding we get

$\sin. (A + B) + \sin. (A - B) = 2 \sin. A . \cos. B \, . \, . \, . \, . \, . \, (P)$

Again, $\cos. (A + B) = \cos. A . \cos. B - \sin. A . \sin. B$

$\cos. (A - B) = \cos. A . \cos. B + \sin. A . \sin. B.$

$\therefore \cos. (A + B) + \cos. (A - B) = 2 \cos. A . \cos. B \, . \, . \, . \, . \, . \, (Q)$

Now in the equations (P) and (Q), for A write 3 A, and for B write 2 A;

Then (P) becomes $\sin. 5A + \sin A = 2 \sin. 3A . \cos. 2A \; . \; (R)$

(Q) „ $\cos. 5A + \cos. A = 2 \cos. 3A . \cos. 2A \; . \; (S)$

Add sin. 3 A to both sides of (R),

$\therefore \sin. A + \sin. 3A + \sin. 5A = 2 \sin. 3A . \cos 2A + \sin. 3A$

$= \sin. 3A (2 \cos. 2A + 1);$

Add cos. 3 A to both sides of (S),

$\therefore \cos. A + \cos. 3A + \cos. 5A = 2 \cos. 3A . \cos. 2A + \cos. 3A$

$= \cos. 3A (2 \cos. 2A + 1),$

$$\text{whence } \frac{\sin. A + \sin. 3A + \sin. 5A}{\cos. A + \cos. 3A + \cos. 5A} = \frac{\sin. 3A (2 \cos. 2A + 1)}{\cos. 3A (2 \cos. 2A + 1)}$$

$$= \tan. 3A.$$

(18) Prove $\sec. A = 1 + \tan. A . \tan. \tfrac{1}{2} A$.

First $\tan. A = \dfrac{2 \tan. \frac{1}{2} A}{1 - \tan.^2 \frac{1}{2} A}$. (See Example 5.)

$$\therefore 1 + \tan. A . \tan. \tfrac{1}{2} A = 1 + \frac{2 \tan. \frac{1}{2} A}{1 - \tan.^2 \frac{1}{2} A} \times \tan. \tfrac{1}{2} A$$

$$= 1 + \frac{2 \tan.^2 \frac{1}{2} A}{1 - \tan.^2 \frac{1}{2} A} = \frac{1 + \tan.^2 \frac{1}{2} A}{1 - \tan.^2 \frac{1}{2} A} = \frac{\sec.^2 \frac{1}{2} A}{1 - \tan.^2 \frac{1}{2} A}$$

$$= \frac{\dfrac{1}{\cos.^2 \frac{1}{2} A}}{1 - \dfrac{\sin.^2 \frac{1}{2} A}{\cos.^2 \frac{1}{2} A}} = \frac{\dfrac{1}{\cos.^2 \frac{1}{2} A}}{\dfrac{\cos.^2 \frac{1}{2} A - \sin.^2 \frac{1}{2} A}{\cos.^2 \frac{1}{2} A}} = \frac{1}{\cos.^2 \frac{1}{2} A - \sin.^2 \frac{1}{2} A}$$

$$= \frac{1}{\cos. (\frac{1}{2} A + \frac{1}{2} A)} (232\ (3)) = \frac{1}{\cos. A} = \sec. A.$$

(19) Prove $\sin. (A + B) . \sin. (A - B) = \sin.^2 A - \sin.^2 B = \cos.^2 B - \cos.^2 A$.

$\sin. (A + B) . \sin. (A - B) = (\sin. A . \cos. B + \cos. A . \sin. B)$

$(\sin. A . \cos. B - \cos. A . \sin. B),$

the right side is of the form $(a + b) (a - b) = a^2 - b^2$;

$\therefore \sin. (A + B) \sin. (A - B) = \sin.^2 A . \cos.^2 B - \cos.^2 A . \sin.^2 B.$

$= \sin.^2 A (1 - \sin.^2 B) - (1 - \sin.^2 A) \sin.^2 B .$

$= \sin.^2 A - \sin.^2 A . \sin.^2 B - \sin.^2 B + \sin.^2 A . \sin.^2 B.$

$= \sin.^2 A - \sin.^2 B.$

Again $\sin.(A+B).\sin.(A-B)=\sin.^2A.\cos.^2B-\cos.^2A.\sin.^2B.$

$$\begin{aligned}&=(1-\cos.^2A)\cos.^2B-\cos.^2A\,(1-\cos.^2B).\\&=\cos.^2B-\cos.^2A\cos.^2B-\cos.^2A+\cos.^2.\,A\cos.^2B.\\&=\cos.^2B-\cos.^2A.\end{aligned}$$

(20) Prove $\tan.^2A-\tan.^2B=\dfrac{\sin.(A+B).\sin.(A-B)}{\cos.^2A.\cos.^2B}.$

$$\tan.^2A-\tan.^2B=(\tan.A+\tan.B)(\tan.A-\tan.B)$$

$$=\left(\frac{\sin.A}{\cos.A}+\frac{\sin.B}{\cos.B}\right)\left(\frac{\sin.A}{\cos.A}-\frac{\sin.B}{\cos.B}\right)=\left(\frac{\sin.A.\cos.B+\cos.A.\sin.B}{\cos.A.\cos.B}\right)$$

$$\times\left(\frac{\sin.A.\cos.B-\cos.A.\sin.B}{\cos.A.\cos.B}\right)=\frac{\sin.(A+B).\sin.(A-B)}{\cos.^2A.\cos.^2B}.$$

THE SINES, COSINES, AND AREA OF A TRIANGLE FOUND IN TERMS OF THE SIDES.

234. *The sides of every triangle are proportional to the sines of the opposite angles.*

In the triangle $A\,B\,C$, figs. IV. and V. let A, B, C, mark the angles, and a, b, c, the sides respectively opposite to them, and from A draw $A\,D$ perpendicular to the base in fig. IV. and to the base produced in fig. V.

Fig. IV.

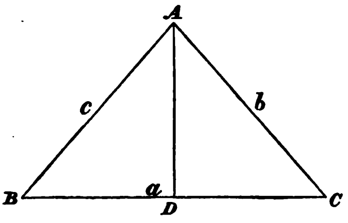
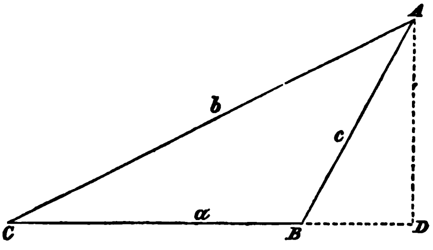


Fig. V.



Then in both figures,

$\sin.B=\dfrac{A\,D}{c}$ (because in V, $\sin.B=\sin.\overline{\pi-B}=\sin.A\,B\,D$)

$$\sin.C=\frac{A\,D}{b}\qquad\therefore\ \frac{\sin.B}{\sin.C}=\frac{\frac{A\,D}{c}}{\frac{A\,D}{b}}=\frac{b}{c}.$$

Similarly, by drawing a perpendicular from B upon $A\,C$,

$$\frac{\sin.A}{\sin.C}=\frac{a}{c},\text{ and }\frac{\sin.A}{\sin.B}=\frac{a}{b}.$$

Otherwise expressed,

$$\text{Sin. } A : \text{sin. } B :: a : b.$$
$$\text{Sin. } A : \text{sin. } C :: a : c.$$
$$\text{Sin. } B : \text{sin. } A :: b : a.$$
$$\text{Sin. } B : \text{sin. } C :: b : c.$$

235. *The cosine of any angle of a triangle in terms of its sides*, may be thus obtained:

In fig. IV. $A C^2 = A B^2 + B C^2 - 2 B C . B D$, by Euclid, book II., prop. 13.

In fig. V. $A C^2 = A B^2 + B C^2 + 2 B C . B D$, by Euclid, book II. prop. 12.

$$\text{Now } \frac{B D}{B A} = \text{cos. } B,$$

$$\begin{aligned} \therefore B D &= B A . \text{cos. } B, \text{ in fig. IV.} \\ &= B A . \text{cos. } A B D, \text{ in fig. V.} \\ &= - B A \text{ cos. } B \text{ (220).} \end{aligned}$$

It therefore follows, that in both cases,

$$b^2 = c^2 + a^2 - 2 a c . \text{cos. } B.$$

$$\therefore \text{cos. } B = \frac{a^2 + c^2 - b^2}{2 a c}; \text{ and similarly,}$$

$$\text{cos. } A = \frac{b^2 + c^2 - a^2}{2 b c};$$

$$\text{cos. } C = \frac{a^2 + b^2 - c^2}{2 a b}.$$

236. *The sine of any angle of a triangle in terms of the sides.* We have, in art. 226,

$$\begin{aligned} \text{Sin.}^2 A &= (1 - \text{cos.}^2 A) \\ &= (1 + \text{cos. } A)(1 - \text{cos. } A.) \end{aligned}$$

Now, by the last article,

$$\begin{aligned} 1 + \text{cos. } A &= 1 + \frac{b^2 + c^2 - a^2}{2 b c} \\ &= \frac{b^2 + 2 b c + c^2 - a^2}{2 b c} \\ &= \frac{(b + c)^2 - a^2}{2 b c} \\ &= \frac{(a + b + c)(b + c - a)}{2 b c} \end{aligned}$$

$$\begin{aligned} 1 - \text{cos. } A &= 1 - \frac{b^2 + c^2 - a^2}{2 b c} \\ &= \frac{a^2 - (b^2 - 2 b c + c^2)}{2 b c} \end{aligned}$$

$$= \frac{a^2 - (b - c)^2}{2\,b\,c}$$

$$= \frac{(a + c - b)(a + b - c)}{2\,b\,c}$$

Let $\frac{a + b + c}{2} = S$, *i. e.*, let S be the semi-perimeter;

Then $\frac{b + c - a}{2} = S - a,$

$$\frac{a + c - b}{2} = S - b,$$

$$\frac{a + b - c}{2} = S - c.$$

But because $S\,.\,(S - a) = \frac{(a + b + c)\,(b + c - a)}{4},$

$$\therefore\ 1 + \cos.\,A = \frac{2\,S\,.\,(S - a)}{b\,c};$$

and because $(S - b)\,.\,(S - c) = \frac{(a + c - b)\,(a + b - c)}{4},$

$$\therefore\ 1 - \cos.\,A = \frac{2\,(S - b)\,.\,(S - c)}{b\,c};$$

and since $\sin.^2 A = (1 + \cos.\,A)\,(1 - \cos.\,A),$

$$\therefore\ \sin.^2 A = \frac{4}{b^2\,c^2}\,.\,S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c),$$

$$\sin.\,A = \frac{2}{b\,c}\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}.$$

By similar deductions,

$$\sin.\,B = \frac{2}{a\,c}\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}$$

$$\sin.\,C = \frac{2}{a\,b}\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}.$$

237. *The area of a triangle in terms of its sides.*

Referring to figs. IV. and V. we find, by Euclid I. 41,

$$\text{Area of triangle } A\,B\,C = \frac{B\,C\,.\,A\,D}{2};$$

$$\text{But } A\,D = A\,B\,.\,\sin.\,B,$$

$$\therefore\ \text{Area of triangle } A\,B\,C = \frac{B\,C\,.\,A\,B\,\sin.\,B}{2} = \frac{a\,c}{2}\,.\,\sin.\,B.$$

$$\text{But } \sin.\,B = \frac{2}{a\,c}\sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)},$$

$$\therefore\ \text{Area of triangle } A\,B\,C = \sqrt{S\,.\,(S - a)\,.\,(S - b)\,.\,(S - c)}.$$

238. To find, in terms of the sides, the sin. $\frac{A}{2}$, cos. $\frac{A}{2}$, and tan. $\frac{A}{2}$.

We have found above, that

$$1 + \cos. A = \frac{2\,S\,.\,(S - a)}{b\,c}, \text{ and } 1 - \cos. A = \frac{2\,(S - b).(S - c)}{b\,c}$$

By 232 (3) we have, $\cos. 2\,A = 2\cos.^2 A - 1 = 1 - 2\sin.^2 A$.

For $2\,A$ substitute A, and for A substitute $\frac{A}{2}$, and

$$\cos. A = 2\cos.^2\frac{A}{2} - 1, \text{ or } = 1 - 2\sin.^2\frac{A}{2};$$

$$\therefore 1 + \cos. A = 2\cos.^2\frac{A}{2}, \text{ and } 1 - \cos. A = 2\sin.^2\frac{A}{2}.$$

$$\therefore 2\cos.^2\frac{A}{2} = \frac{2\,S\,.\,(S - a)}{b\,c}, \text{ and } 2\sin.^2\frac{A}{2} = \frac{2\,.\,(S - b)\,.\,(S - c)}{b\,c}$$

$$\therefore \sin.\frac{A}{2} = \sqrt{\frac{(S - b)\,.\,(S - c)}{b\,c}}, \text{ and } \cos.\frac{A}{2} = \sqrt{\frac{S\,.\,(S - a)}{b\,c}}$$

In the same way we obtain,

$$\sin.\frac{B}{2} = \sqrt{\frac{(S - a)\,.\,(S - c)}{a\,c}}; \quad \cos.\frac{B}{2} = \sqrt{\frac{S\,.\,(S - b)}{a\,c}}$$

$$\sin.\frac{C}{2} = \sqrt{\frac{(S - a)\,.\,(S - b)}{a\,b}}; \quad \cos.\frac{C}{2} = \sqrt{\frac{S\,.\,(S - c)}{a\,b}}$$

$$\text{And } \tan.\frac{A}{2} = \frac{\sin.\frac{A}{2}}{\cos.\frac{A}{2}} = \sqrt{\frac{(S - b)\,.\,(S - c)}{S\,.\,(S - a)}};$$

$$\tan.\frac{B}{2} = \sqrt{\frac{(S - a)\,.\,(S - c)}{S\,.\,(S - b)}}; \quad \tan.\frac{C}{2} = \sqrt{\frac{(S - a)\,.\,(S - b)}{S\,.\,(S - c)}}.$$

SOLUTION OF TRIANGLES.

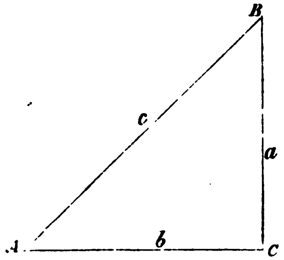
The parts of a triangle are six, the three sides and the three angles, any three of which (except the three angles) being given, we can find the rest by means of the rules deduced in the preceding chapters.

Formulaic Solution of Triangles.

RIGHT ANGLED TRIANGLES.

239. By the previous definitions, the following equations are obviously true:—

FIG. VI.



$$\sin. A = \frac{a}{c} = \cos. B \therefore a = c \,.\, \sin. A = c \,.\, \cos. B$$

$$\cos. A = \frac{b}{c} = \sin. B \therefore b = c \,.\, \cos. A = c \,.\, \sin. B$$

$$\tan. A = \frac{a}{b} = \cot. B \therefore a = b \,.\, \tan. A = b \,.\, \cot. B$$

$$c = \sqrt{a^2 + b^2}, \therefore b = \sqrt{c^2 - a^2} = \sqrt{c - a} \,.\, \sqrt{c + a},$$

$$\text{and } a = \sqrt{c - b} \,.\, \sqrt{b + c}.$$

OBLIQUE ANGLED TRIANGLES.

First Case.

240. The solution of a triangle, when the three sides a, b, c, are given (figs. IV. and V.).

$$\text{Let } \frac{a + b + c}{2} = S, \text{ as before,}$$

$$\text{and first obtain } M = \sqrt{\frac{(S - a) \,.\, (S - b) \,.\, (S - c)}{S}};$$

then find the three angles by means of the following formulæ, which are very obvious deductions from the values of the same functions found in section 238.

$$\tan. \frac{A}{2} = \frac{M}{S - a}, \tan. \frac{B}{2} = \frac{M}{S - b}, \tan. \frac{C}{2} = \frac{M}{S - c}.$$

The above formula is very convenient when the three angles are required; but for finding the value of one angle, any of the formulæ of the angle and the sine above deduced may be employed. For the sine, cosine, or tangent of half the angle, it must be here and also generally remembered, that when the angle is near 90°, the sine, tangent, or secant should not be used; nor the cosine, cotangent, or cosecant, when the angle is very small; because in each case the functions named vary with the angle less than their cofunctions.

Second Case.

241. Given one side a, and two angles, A and B, to find the rest.

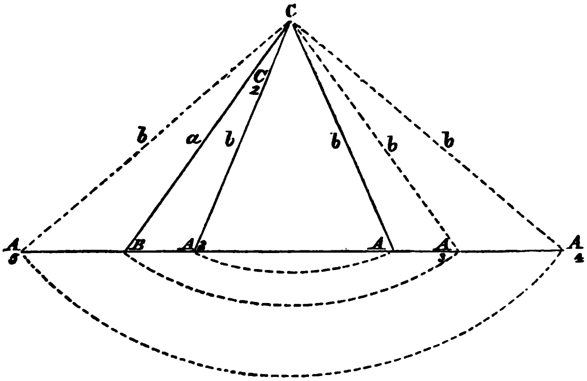
The third angle $C = 180° -$ angles $(A + B)$, and it has been proved that $\frac{b}{a} = \frac{\sin. B}{\sin. A}$; also that $\frac{c}{a} = \frac{\sin. C}{\sin. A}$;

$$\therefore b = a\frac{\sin. B}{\sin. A}, \text{ and } c = a\frac{\sin. C}{\sin. A}.$$

Third Case.

242. Given two sides a b, and an angle B not included, to find the rest. This is usually called the *ambiguous case.*

FIG. VII.



If B be acute, and $b < a$, we have two triangles indicated, $A\ B\ C$ and $A_2\ B\ C$, and therefore two angles, A and A_2, supplemental to each other; and since $\sin. B = \sin. (180 - B)$, we have two solutions, and no means of determining which of these angles is meant.

If B be acute, and $b = a$, as in $A_3\ B\ C$, then $A = B$, and if $b > a$, as in $A_4\ B\ C$, then $A < B$. Lastly, if B be obtuse, as in $A_5\ B\ C$, A is acute, and in none of these cases, is there any ambiguity.

When the ambiguous case is given for solution, the ratio of the required angle to a right angle is generally given also; where this is not done, both angles should be found as follows:

$$\text{First } \because \frac{\sin. A}{\sin. B} = \frac{a}{b} \quad \therefore \sin. A = \sin. B\frac{a}{b};$$

$$C = 180° - (A + B).$$

$$\text{And by the second case } c = a \,.\, \frac{\sin. C}{\sin. A}.$$

Fourth Case.

243. Given two sides, a b, and the included angle C to find the other two angles and remaining side.

We have found that $\frac{a}{b} = \frac{\sin. A}{\sin. B}$,

$$\text{First } A = \frac{A + B}{2} + \frac{A - B}{2} \quad \ldots\ldots\ldots\ldots \quad (Q)$$

$$B = \frac{A + B}{2} - \frac{A - B}{2} \quad \ldots\ldots\ldots\ldots \quad (R)$$

$$\text{Then } \frac{a + b}{a - b} = \frac{\sin. A + \sin. B}{\sin. A - \sin. B} \text{ (p. 146, Ex. 17)} \quad . \ . \ (P)$$

$$= \frac{2 \sin. \frac{A + B}{2} . \cos. \frac{A - B}{2}}{2 \cos. \frac{A + B}{2} . \sin. \frac{A - B}{2}}; \text{ dividing num. and}$$

$$\text{denom. by } 2 \cos. \frac{A + B}{2} \ . \ \cos. \frac{A - B}{2}$$

$$= \frac{\tan. \frac{A + B}{2}}{\tan. \frac{A - B}{2}}.$$

$$\text{Now } C = 180^\circ - \overline{A + B} \ \therefore \frac{C}{2} = 90^\circ - \frac{A + B}{2}$$

$$\therefore \cot. \frac{C}{2} = \tan. \frac{A + B}{2}$$

$$\text{and } \frac{a + b}{a - b} = \frac{\cot. \frac{C}{2}}{\tan. \frac{A - B}{2}}$$

$$\therefore \tan. \frac{A - B}{2} = \frac{a - b}{a + b} . \cot. \frac{C}{2}.$$

By the last formula, having two sides, a b, and an angle C given, the other two angles and remaining side may be easily found.

$$\text{First } \frac{A + B}{2} = 90^\circ - \frac{C}{2},$$

and $\frac{A - B}{2}$ can be found by the above formula.

Then A and B can be found from (Q) and (R), and as before shown,

$$c = a \frac{\sin. C}{\sin. A}.$$

244. But in order to obtain c without finding the angles, A or B, the two following methods are usually given.

First Method.

$$c = \sqrt{a^2 + b^2 - 2 \, a \, b \cos. C}$$

$$\text{and } \cos. C = 2 \cos.^2 \frac{C}{2} - 1. \quad (238.)$$

$$\therefore c^2 = a^2 + b^2 - 4\,a\,b\,.\cos.^2\frac{C}{2} + 2\,a\,b$$

$$= (a + b)^2 - 4\,a\,b\,.\cos.^2\frac{C}{2}$$

$$= (a + b)^2\left\{1 - \frac{4\,a\,b\,.\cos.^2\frac{C}{2}}{(a + b)^2}\right\}$$

Suppose $\dfrac{4\,a\,b\,.\cos.^2\frac{C}{2}}{(a + b)^2} = \sin.^2\theta$,

then $c^2 = (a + b)^2\,(1 - \sin.^2\theta) = (a + b)^2\cos.^2\theta$,
and $c = (a + b)\cos.\,\theta$.

Second Method.

As $\cos.\,C = 1 - 2\sin.^2\frac{C}{2}$ (238)

$$c^2 = a^2 + b^2 - 2\,a\,b + 4\,a\,b\sin.^2\frac{C}{2}$$

$$= (a - b)^2 + 4\,a\,b\,.\sin.^2\frac{C}{2}$$

$$= (a - b)^2\left\{1 + \frac{4\,a\,b\sin.^2\frac{C}{2}}{(a - b)^2}\right\}$$

Suppose $\dfrac{4\,a\,b\,.\sin^2\frac{C}{2}}{(a - b)^2} = \tan.^2\theta$.

Then $c^2 = (a - b)^2\,(1 + \tan.^2\theta)$
$= (a - b)^2\sec.^2\theta$.
And $c = (a - b)\,.\sec.\,\theta$.

Numerical Solution of Triangles.

RIGHT ANGLED TRIANGLES.

The cases which may be proposed for solution, in right-angled triangles, are the following :—

First Case.

245. Given c and a or b; required the rest.

By 239, $\sin.\,A = \frac{a}{c}$; or (if b be given) $\cos.\,A = \frac{b}{c}$.
and $b = c\cos.\,A$; or (if b be given) $a = c\,.\sin.\,A$
$\therefore \log._t\sin.\,A\,. = 10 + \log.\,a - \log.\,c$
and $\log.\,b = \log.\,c + \log._t\cos.\,A - 10$.

If b be given, $\log._t \cos. A = 10 + \log. b - \log. c$
and $\log. a = \log. c + \log._t \sin. A - 10$.

In both cases, the remaining $\angle = 90^\circ - \angle$ found.

Example. Given $c = 4184$
$b = 2632$ to find the rest.

Here, $\log._t \cos. A = 10 + \log. b - \log. c$.

$$
\begin{aligned}
\log. b &= 3{\cdot}4202859 \\
\text{,,}\quad c &= 3{\cdot}6215917 \\
\hline
&\overline{1}{\cdot}7986942 \ \log. \cos. 51^\circ\ 1'\ 8\tfrac{1}{3}''
\end{aligned}
$$

The logarithmic formula for deducing a, as given above, is, $\log. a = \log. c + \log._t \sin. A - 10$; but as any of the formulæ given in article 239 may be used, we take this one, $a = b \tan. A$.

$$
\begin{aligned}
\therefore \log. a &= \log. b + \log._t \tan. A - 10 \\
&= \log. \tan. 51^\circ\ 1' = {\cdot}0918891 \\
&\qquad\qquad 8\tfrac{1}{3}'' = \phantom{{\cdot}0918}359 \\
\hline
&\qquad\qquad {\cdot}0919250 \\
&\qquad \log. b = 3{\cdot}4202859 \\
\hline
&\qquad\qquad 3{\cdot}5122109 = \log. \text{ of } 3252{\cdot}452
\end{aligned}
$$

$$
\begin{aligned}
A &= 51^\circ\ 1'\ 8\tfrac{1}{3}'' \\
B &= 38^\circ\ 58'\ 51\tfrac{2}{3}'' \\
a &= 3252{\cdot}452
\end{aligned}
$$

246. The above solutions give a by means of the angle which is first found; but the side a or b may be found without knowing the angles.

For $b = \sqrt{c - a} \cdot \sqrt{c + a}$ or (if b be given) $a = \sqrt{c - b} \cdot \sqrt{c + b}$.

$\therefore$ log. of remaining side $= \frac{1}{2}(\log. \overline{c + a \text{ or } b} + \log. \overline{c - a \text{ or } b})$

Ex. Given $c = 128{\cdot}4327$
$a = 110{\cdot}0951$

$\therefore c + a = 238{\cdot}5278 \qquad c - a = 18{\cdot}3376$

$$
\begin{aligned}
\log. (c + a) &= 2{\cdot}3775390 \\
\log. (c - a) &= 1{\cdot}2633425 \\
\hline
&2)3{\cdot}6408815 \\
\hline
&\quad 1{\cdot}8204408 = \log. \text{ of } 66{\cdot}13644
\end{aligned}
$$

$\therefore b = 66{\cdot}13644$.

Second Case.

247. Given c, and A or B; required the rest.

First, the remaining $\angle = 90^\circ -$ given $\angle$;
and $\because a = c \,.\, \sin.\, A$; and $b = c \,.\, \cos.\, A$;
$\therefore \log.\, a = \log.\, c + \log._t \sin.\, A - 10$,
$\log.\, b = \log.\, c + \log._t \cos.\, A - 10$.

Ex. Given $c = 170\cdot325$,
$A = 44^\circ\ 1'\ 10''$.
Log. $c = 2\cdot2312783$
log. cos. $A = \bar{1}\cdot8567918$

$2\cdot0880701 =$ log. of $122\cdot4814$.

Log. $c = 2\cdot2312783$
log. sin. $A = \bar{1}\cdot8419239$

$2\cdot0732022 =$ log. of $118\cdot3592$

$\therefore B = 90^\circ - 44^\circ\ 1'\ 10'' = 45^\circ\ 58'\ 50''$,
$a = 118\cdot087$, $b = 122\cdot4814$.

Third Case.

248. Given a or b, and A or B; required the rest.

Remaining $\angle$ = 90o — given $\angle$;

$$\text{and} \because \sin.\, A = \frac{a}{c},\ \tan.\, A = \frac{a}{b},\ \text{and } \tan.\, B = \frac{b}{a};$$

$$\therefore c = \frac{a}{\sin.\, A},\ b = \frac{a}{\tan.\, A},\ \text{and (if } b \text{ be given) } a = \frac{b}{\tan.\, B}.$$

$$\therefore \log.\, c = \begin{cases} \log.\, a - \log._t \sin.\, A + 10 \\ \log.\, b - \log._t \sin.\, B + 10 \end{cases}$$

$$\log.\, b = \log.\, a - \log._t \tan.\, A + 10$$
$$\log.\, a = \log.\, b - \log._t \tan.\, B + 10.$$

Ex. Given $b = 469\cdot34$
$A = 51^\circ\ 26'\ 17''$

First, $B = \overline{90^\circ - 51^\circ\ 26'\ 17''} = 38^\circ\ 33'\ 43''$;
log. $b = 2\cdot6714876$
log. tan. $B = \bar{1}\cdot9015688$

$2\cdot7699188 =$ log. of $588\cdot7335$
$\therefore a = 588\cdot7335$.

$$\begin{aligned} \text{Log. tan. } A &= \cdot 0984312 \\ \log. b &= 2\cdot 6714876 \\ \hline \log. a &= 2\cdot 7699188 = \text{log. of } 588\cdot 7335, \\ \text{log. sin. } A &= \bar{1}\cdot 8931705 \\ \hline \log. c &= 2\cdot 8767483 = \text{log. of } 752\cdot 919. \end{aligned}$$

Fourth Case.

249. Given a and b, to find the rest.

$$\because \tan. A = \frac{a}{b},$$

$$\therefore \log._t \tan. A = 10 + \log. a - \log. b,$$

$$\text{and } \angle B = 90^\circ - A;$$

$$\text{also, } \sin. A = \frac{a}{c}, \text{ and } \cos. A = \frac{b}{c},$$

$$\therefore c = \frac{a}{\sin. A}, = \frac{b}{\sin. B};$$

$$\therefore \log. c = \begin{cases} \log. a - \log._t \overline{\sin. A - 10} \\ \log. b - \log._t \overline{\sin. B - 10} \end{cases}$$

Ex. Given $a = 101$,

$b = 103$.

$$\begin{aligned} \text{Log. } a &= 2\cdot 0043214 \\ \log. b &= 2\cdot 0128372 \\ \hline & \bar{1}\cdot 9914842 = \text{log. of tan. } 44^\circ\, 26'\, 18'' \\ \therefore A &= 44^\circ\, 26'\, 18'' \\ B &= \overline{90^\circ - 44^\circ\, 26'\, 18''} = 45^\circ\, 33'\, 42'' \\ \log. a &= 2\cdot 0043214 \\ \text{log. sin. } A &= \bar{1}\cdot 8451856 \\ \hline & 2\cdot 1591358 = \text{log of } 144\cdot 257 \\ & \therefore c = 144\cdot 257. \end{aligned}$$

OBLIQUE ANGLED TRIANGLES.

First Case.

250. Given a, b, and c; required the angles.

The formulæ proved in art. 240 are,

$$S = \frac{a + b + c}{2}; \quad M = \sqrt{\frac{\overline{S - a} \,.\, \overline{S - b} \,.\, \overline{S - c}}{S}}; \text{ then}$$

$$\tan. \frac{A}{2} = \frac{M}{S - a}, \quad \tan. \frac{B}{2} = \frac{M}{S - b}, \quad \tan. \frac{C}{2} = \frac{M}{S - c}.$$

Example. Given

$$\begin{array}{r}
a = 785{\cdot}8 \\
b = 720{\cdot}8 \\
c = 643{\cdot}2 \\
\hline
2)2149{\cdot}8 \\
\hline
1074{\cdot}9 = S \\
289{\cdot}1 = S - a \\
354{\cdot}1 = S - b \\
431{\cdot}7 = S - c
\end{array}$$

251. In working from the above formulæ, the student should perform the subtractions, as has been done above, without writing out the quantities twice.

To find M, and hence A, B, and C.

First, $\log. M = \dfrac{\log. (S - a) + \log. (S - b) + \log. (S - c) - \log. S}{2}$

then, $\log._t \tan. \dfrac{A}{2} = 10 + \log. M - \log. (S - a)$,

$\log._t \tan. \dfrac{B}{2} = 10 + \log. M - \log. (S - b)$,

$\log._t \tan. \dfrac{C}{2} = 10 + \log. M - \log. (S - c)$.

$$\begin{array}{r}
\text{Log.}\,(S - a) = 2{\cdot}4610481 \\
\log.\,(S - b) = 2{\cdot}5491259 \\
\log.\,(S - c) = 2{\cdot}6351820 \\
\hline
7{\cdot}6453560 \\
\log.\,S = 3{\cdot}0313681 \\
\hline
2)4{\cdot}6139879 \\
\hline
2{\cdot}3069939
\end{array} = \log.\,M.$$

$$\begin{array}{r}
\text{Log.}\,M = 2{\cdot}3069939 \\
\log.\,(S - a) = 2{\cdot}4610481 \\
\hline
\bar{1}{\cdot}8459458 = \log.\,\tan.\,35^\circ\,2'\,40{\cdot}5'' \\
{\cdot}8457644 \\
\hline
1814 \\
60 \\
\hline
2688)108840(40'' \\
10752 \\
\hline
1320
\end{array}$$

$$\begin{array}{rl}
\text{Log. } M = & 2{\cdot}3069939 \\
\text{log. } (S - b) = & 2{\cdot}5491259 \\
\hline
& \bar{1}{\cdot}7578680 = \text{log. tan. } 29^\circ\ 47'\ 47'' \\
& \phantom{\bar{1}{\cdot}}7576383 \\
\hline
& \phantom{\bar{1}{\cdot}757}2297 \\
& \phantom{\bar{1}{\cdot}7578}60 \\
\hline
& 293{,}0)13782{,}0(47'' \\
& \phantom{293{,}0)}1172 \\
\hline
& \phantom{293{,}0)1}2062 \\
& \phantom{293{,}0)1}2051 \\
\hline
& \phantom{293{,}0)123}11
\end{array}$$

$$\begin{array}{rl}
\text{Log. } M = & 2{\cdot}3069939 \\
\text{log. } (S - c) = & 2{\cdot}6351820 \\
\hline
& \bar{1}{\cdot}6718119 = \text{log. tan. } 25^\circ\ 9'\ 32{\cdot}5'' \\
& \phantom{\bar{1}{\cdot}}6716345 \\
\hline
& \phantom{\bar{1}{\cdot}671}1774 \\
& \phantom{\bar{1}{\cdot}6718}60 \\
\hline
& 3283)106440(32 \\
& 9849 \\
\hline
& 7950 \\
& 6566 \\
\hline
& 1384
\end{array}$$

$$\begin{array}{lll}
\therefore \dfrac{A}{2} = 35^\circ\ 2'\ 40{\cdot}5'' & \qquad & A = 70^\circ\ 5'\ 21'' \\
\dfrac{B}{2} = 29^\circ\ 47'\ 47'' & & B = 59^\circ\ 35'\ 34'' \\
\dfrac{C}{2} = 25^\circ\ 9'\ 32{\cdot}5'' & & C = 50^\circ\ 19'\ 5'' \\
\hline
\phantom{\dfrac{C}{2} = }90^\circ & & 180^\circ
\end{array}$$

Second Case.

252. Given one side a, and two angles A and B, to find the rest.

First, $C = 180^\circ - (A + B)$,

then $b = a \cdot \dfrac{\text{sin. } B}{\text{sin. } A}$, and $c = a \cdot \dfrac{\text{sin. } C}{\text{sin. } A}$

$\therefore$ log. b = log. a + log. sin. B − log. sin. A,

log. c = log. a + log. sin. C − log. sin. A.

Ex. Given $a = 15\cdot236$

$$\begin{aligned} A &= 89^\circ\ \ 9'\ 23\cdot54'' \\ B &= 54^\circ\ 33'\ 25\cdot12'' \end{aligned}$$

First, $C = 180^\circ - (A + B) = 36^\circ\ 17'\ 11\cdot34''$.

$$\begin{array}{rl}
\text{Log. } a = & 1\cdot1828710 \\
\text{log. sin. } B = & \bar{1}\cdot9109937 \\ \cline{2-2}
 & 1\cdot0938647 \\
\text{log. sin. } A = & \bar{1}\cdot9999530 \\ \cline{2-2}
 & 1\cdot0939117 = \text{log. of } 12\cdot414 = b \\
\text{Log. } a = & 1\cdot1828710 \\
\text{log. sin. } C = & \bar{1}\cdot7721918 \\ \cline{2-2}
 & \cdot9550628 \\
 & \bar{1}\cdot9999530 \\ \cline{2-2}
 & \cdot9551098 = \text{log. of } 9\cdot01799 = c
\end{array}$$

$\therefore b = 12\cdot414$ and $c = 9\cdot018$.

Third Case.

253. Given two sides a, b, and an angle, B not included, to find the rest.

On referring to fig. VII. and art. 242, we find

$$\sin. A = \sin. B\,\frac{a}{b}; \quad \begin{aligned} C &= 180^\circ - \overline{A + B} \\ C_2 &= 180^\circ - \overline{A_2 + B} \end{aligned}$$

$$c = a\,\frac{\sin. C}{\sin. A}. \quad c_2 = a\,\frac{\sin. C_2}{\sin. A_2}.$$

Here by c and c_2 the sides $B\,A$ and $B\,A_2$ are meant; and by C and C_2, their opposite angles.

First, $\log. \sin. A = \log. a + \log. \sin. B - \log. b$,

and $\log. c = \log. a + \log. \sin. C - \log. \sin. A$,

$\log. c_2 = \log. a + \log. \sin. C_2 - \log. \sin. A_2$.

Ex. Given $a = 234$

$$\begin{aligned} b &= 159 \\ B &= 27^\circ\ 13'. \end{aligned}$$

$$\begin{array}{rl}
\text{Log. sin. } B = & \bar{1}\cdot6602550 \\
\text{log. } a = & 2\cdot3692159 \\ \cline{2-2}
 & 2\cdot0294709 \\
\text{log. } b = & 2\cdot2013971 \\ \cline{2-2}
 & \bar{1}\cdot8280738 = \text{log. of sin. } 42^\circ\ 18'\ 21\cdot91''
\end{array}$$

$$\therefore A = 42^\circ\ 18'\ 21{\cdot}91'' : A_2 = 137^\circ\ 41'\ 38{\cdot}09''$$

$$C = 180^\circ - (42^\circ\ 18'\ 21{\cdot}91'' + 27^\circ\ 13') = 110^\circ\ 28'\ 38{\cdot}09''$$

$$C_2 = 180^\circ - (137^\circ\ 41'\ 38{\cdot}09'' + 27^\circ\ 13') = 15^\circ\ 5'\ 21{\cdot}91''$$

$$\begin{array}{rl}
\log.\ a = & 2{\cdot}3692159 \\
\log.\ \sin.\ C = & \bar{1}{\cdot}9716520 \\
\hline
& 2{\cdot}3408679 \\
\log.\ \sin.\ A = & \bar{1}{\cdot}8280728 \\
\hline
& 2{\cdot}5127951 = \log.\ \text{of}\ 325{\cdot}683 = c \\
\log.\ a = & 2{\cdot}3692159 \\
\log.\ \sin.\ C_2 = & \bar{1}{\cdot}4156471 \\
\hline
& 1{\cdot}7848630 \\
\log.\ \sin.\ A_2 = & \bar{1}{\cdot}8280728 \\
\hline
& 1{\cdot}9567902 = \log.\ \text{of}\ 90{\cdot}5294 = c_2
\end{array}$$

∴ if the solution of the triangle $B\ A\ C$ be required,

$A = 42^\circ\ 18'\ 21{\cdot}91''$; $C = 110^\circ\ 28'\ 38{\cdot}09''$

$c = 325{\cdot}6830.$

But if the triangle $B\ A_2\ C_2$ be given for solution,

$A_2 = 137^\circ\ 41'\ 38{\cdot}09''$; $C_2 = 15^\circ\ 5'\ 21{\cdot}91''$

$c_2 = 90{\cdot}5294.$

Fourth Case.

254. Given two sides a, b, and the included angle C, to find the other two angles and remaining side.

$$\tan.\frac{A-B}{2} = \frac{a-b}{a+b}.\cot.\frac{C}{2};\ \text{and}\ \frac{A+B}{2} = 90^\circ - \frac{C}{2}.$$

$$A = \frac{A+B}{2} + \frac{A-B}{2};\ \text{and}\ B = \frac{A+B}{2} - \frac{A-B}{2}.$$

$$c = a\frac{\sin.\ C}{\sin.\ A}.$$

The above written for logarithmic computation is

$$\log.\ \tan.\frac{A-B}{2} = \log.\ \overline{a-b} + \log.\ \cot.\frac{C}{2} - \log.\ \overline{a+b}$$

$$\log.\ c = \log.\ \sin.\ C + \log.\ a - \log.\ \sin.\ A.$$

Ex. Given $a = 85{\cdot}63$

$b = 78{\cdot}21$

$C = 48^\circ\ 24'$

$$\text{First } A + B = 180^\circ - 48^\circ\, 24' = 131^\circ\, 36'$$

$$\therefore \frac{A+B}{2} = 65^\circ\, 48' \text{ and } \frac{C}{2} = 24^\circ\, 12'$$

$$a - b = 7{\cdot}42 \qquad \frac{C}{2} = 24^\circ\, 12'$$

$$a + b = 163{\cdot}84 \qquad \frac{A+B}{2} = 65^\circ\, 48'$$

$$\begin{aligned}
\text{log. }(a-b) &= {\cdot}8704039 \\
\text{log. cot. }\frac{C}{2} &= \underline{{\cdot}3473497} \\
& 1{\cdot}2177536 \\
\text{log. }(a+b) &= \underline{2{\cdot}2144199}
\end{aligned}$$

$$\bar{1}{\cdot}0033337 = \text{log. of tan. } \frac{A-B}{2} = 5^\circ\, 45'\, 15{\cdot}5''$$

$$\therefore \frac{1}{2}(A+B) = 65^\circ\ 48'$$

$$\text{and } \frac{1}{2}(A-B) = \underline{5^\circ\ 45'\ 15{\cdot}5''}$$

$$71^\circ\ 33'\ 15{\cdot}5'' = A.$$

$$\overline{60^\circ\ \ 2'\ 44{\cdot}5''} = B.$$

$$\begin{aligned}
\text{Log. } a &= 1{\cdot}9326259 \\
\text{log. sin. } C &= \underline{\bar{1}{\cdot}8737844} \\
& 1{\cdot}8064103 \\
\text{log. sin. } A &= \underline{\bar{1}{\cdot}9770941} \\
& 1{\cdot}8293162 = \text{log. of } 67{\cdot}50194 \\
\therefore\ c &= 67{\cdot}50194.
\end{aligned}$$

When the side "c" is required without making use of an angle.

First Method.

$$\text{sin. }\theta = \frac{2 \cos. \frac{C}{2} \sqrt{a\,b}}{a + b}$$

$$\text{and } c = (a + b) \cos. \theta.$$

$$\therefore \log._t \text{sin.}\,\theta = \tfrac{1}{2}(\text{log. } a + \text{log. } b) + \log._t \cos. \frac{C}{2} + \text{log.} 2 - \text{log.}(a+b)$$

$$\text{and log. } c = \text{log. } \overline{a+b} + \log._t \cos. \theta - 10.$$

Ex. As before let $a = 85{\cdot}63$
$b = 78{\cdot}21$

$$\begin{array}{rl}
\log.\ a = & 1{\cdot}9326259 \\
\log.\ b = & 1{\cdot}8932623 \\
\cline{2-2}
2) & 3{\cdot}8258882 \\
\cline{2-2}
 & 1{\cdot}9129441 \\
\log.\ \cos.\ \frac{C}{2} = & \bar{1}{\cdot}9600520 \\
\log.\ 2 = & {\cdot}3010300 \\
\cline{2-2}
 & 2{\cdot}1740261 \\
\log.\ (a+b) = & 2{\cdot}2144199 \\
\cline{2-2}
 & \bar{1}{\cdot}9596062 = \log.\ \sin.\ 65^\circ\ 40'\ 10{\cdot}3''
\end{array}$$

$\therefore\ \theta = 65^\circ\ 40'\ 10{\cdot}3''$.

$$\begin{array}{rl}
\log.\ (a+b) = & 2{\cdot}2144199 \\
\log.\ \cos.\ \theta = & \bar{1}{\cdot}6148962 \\
\cline{2-2}
 & 1{\cdot}8293161 = \log.\ \text{of}\ 67{\cdot}50194.
\end{array}$$

$\therefore\ c = 67{\cdot}50194$ as before.

Second Method.

$$\text{Tan.}\ \theta = \frac{2\sin.\frac{C}{2}\sqrt{a\,b}}{a-b}\ \text{and}\ c = (a-b)\sec.\ \theta.$$

$$\therefore \log._t \tan.\ \theta = \tfrac{1}{2}(\log.\ a + \log.\ b) + \log.\ 2 + \log._t \sin.\ \frac{C}{2} - \log.\ (a-b),$$

$$\text{and}\ \log.\ c = \log.\ (a-b) - 10 + \log._t \sec.\ \theta.$$

$$\begin{array}{rl}
\log.\ a = & 1{\cdot}9326259 \\
\log.\ b = & 1{\cdot}8932623 \\
\cline{2-2}
2) & 3{\cdot}8258882 \\
\cline{2-2}
 & 1{\cdot}9129441 \\
\log.\ 2 = & {\cdot}3010300 \\
\log.\ \sin.\ \frac{C}{2} = & \bar{1}{\cdot}6127023 \\
\cline{2-2}
 & 1{\cdot}8266764 \\
\log.\ \overline{a-b} = & {\cdot}8704039 \\
\cline{2-2}
\log.\ \tan.\ \theta = & 0{\cdot}9562725 = \log.\ \tan.\ 83^\circ\ 41'\ 20{\cdot}87'' \\
\log.\ \overline{a-b} = & {\cdot}8704039 \\
\log.\ \sec.\ \theta = & {\cdot}9589123 \\
\cline{2-2}
\log.\ c = & 1{\cdot}8293162 = \log.\ \text{of}\ 67{\cdot}50194.
\end{array}$$

$c = 67{\cdot}50194$ as before.

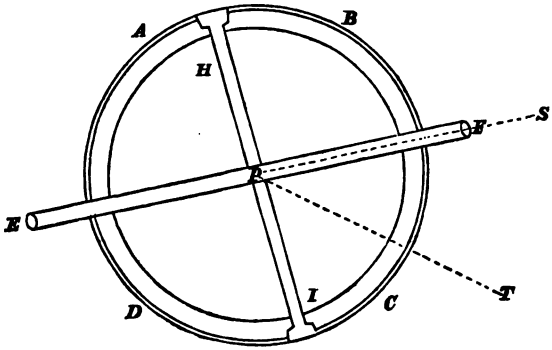
ON THE MEASUREMENT OF HEIGHTS AND DISTANCES.

255. In the operations necessary for this purpose, the length of at least one line must be ascertained by actual admeasurement, and the magnitude of certain angles by observation.

The former may be effected without difficulty when no great accuracy is required, by means of a string, chain, or pole, of a given length; when extreme nicety is essential, as in extensive trigonometrical surveys, many precautions must be taken, which, however, it is not our object particularly to point out.

We shall proceed to show generally how the angular distance of two points may be observed.

Fig. VIII.



Let *A B C D* be a circle divided into 360 degrees, and connected with its centre by spokes or rays firmly united to its circumference or *limb*. At the centre let a circular hole be pierced, in which shall move a pivot exactly fitting it, carrying a tube whose axis, *E F*, is exactly parallel to the plane of the circle; and also the two arms, *H I*, at right angles to it, and forming one piece with the tube and the axis, so that the motion of the axis on the centre shall carry the tube and arms smoothly round the circle to be arrested and fixed at any point we please, by a contrivance called a clamp. Suppose now we would measure the angular interval between two fixed objects, *S T*. The plane of the circle must first be adjusted so as to pass through them both. This done, let the axis, *E F*, of the tube be directed to one of them, *S*, and clamped. Then will a mark on the arm, *H*, point either exactly to some one of the divisions on the limb, or between two of them adjacent.

In the former case, the division must be noted as *the reading* of the arm, *H*.

In the latter, the fractional part of one whole interval between the consecutive divisions, by which the mark on H surpasses the last inferior division, must be estimated or measured by some mechanical or optical means.

The division and fractional part thus noted, and reduced into degrees, minutes, and seconds, is to be set down as the *reading of the limb*, corresponding to that position of the tube, $E\ F$, where it points to the object S. The same must then be done for the object T. Then if the less of the readings be subtracted from the greater, *their difference* will be the angular interval between S and T, as seen from the centre of the circle.

256. The most common instruments for measuring angles in terrestrial observation, are the quadrant, repeating circle, and theodolite, in all of which the perfect circularity and exact graduation of the rim are most essential; and though the graduating is executed in the present day with wonderful precision, it is a matter of too much practical difficulty not to admit of some correction by methods, the principles of which are independent of mere mechanical skill.

The most important is the principle of *repetition*, by which any error in the graduation and in the *reading off* of the number of degrees, &c., to which a single observation is liable, is divided among many *repeated* observations, so that by a sufficient number of repetitions, the required angle, as far as it depends on the above circumstances, can be obtained to any assignable degree of accuracy. This is easily explained.

When the axis of the tube has been directed first to S and then to T, as above described, let it be fixed to the rim, and then be made to move by the motion of the rim about its axis, till it comes again into the direction $P\ S$. Let $E\ F$ be then unfastened from the rim, and brought again into the direction $P\ T$; the graduation corresponding to this position of $E\ F$, will denote *twice* the required angle $S\ P\ T$; and if s'' be the error arising from the graduation or *reading off* after the second observation, and A denote the angle read off in degrees, minutes, &c., then,

$$2\,S\,P\,T = A + s'',$$

$$S\,P\,T = \frac{A}{2} + \frac{s''}{2},$$

or the error in the observed value of $S\ P\ T$ will be $\frac{s''}{2}$. For three observations the error would have been $\frac{s''}{3}$; for n observations $\frac{s''}{n}$. The reading off is only once necessary, *viz.*, after the last observation. It is in the application of this principle that the peculiarity of the *repeating circle* consists.

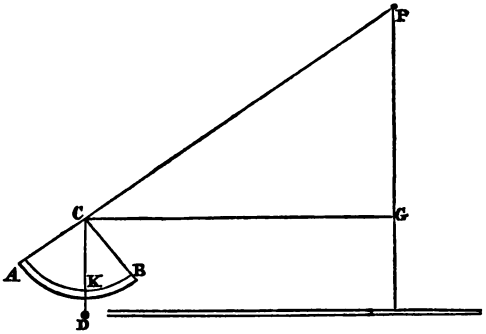
Another principle of correction for erroneous graduation and reading off, consists in taking the *mean* of several readings off on different parts of the rim as the true reading.

257. A quadrant is simply a quarter of such a circle as that above described, graduated from 0° to 90°. It has the disadvantage of offering no practical facility for the application of the above principles of correction.

In the theodolite the graduated circle is horizontal when properly adjusted, the instrument being intended for the observation of horizontal angles only.

To ascertain the height of an object, it is most convenient to observe its angular altitude, or the angle which a line from the object to the observer's eye makes with a horizontal line. The horizontal line being seldom exactly known, it is usual to make use of a plumb-line, which determines the position of a line perpendicular to the horizon.

FIG. IX.



Thus, if P be the object, let the edge of the quadrant be directed to P; then if $C\,D$ be the plumb-line, the graduation of $B\,K$ will give the angle $P\,C\,G$, $C\,G$ being horizontal.

258. To find the height and distance of an inaccessible object on a horizontal plane.

Let the angle $A\,C\,D$ (Fig. X.) observed at the station $C = \alpha$. Measure $C\,B$ in the direction of the object, and let it $= a$; then observe the angle $A\,B\,D = \beta$. Let the height $A\,D = y$, and the distance $B\,D = x$.

$$\text{Now } \frac{A\,B}{B\,C} = \frac{\sin.\,A\,C\,B}{\sin.\,B\,A\,C} = \frac{\sin.\,\alpha}{\sin.\,(\beta - \alpha)} = \sin.\,\alpha\,.\,\text{cosec.}\,(\beta - \alpha).$$

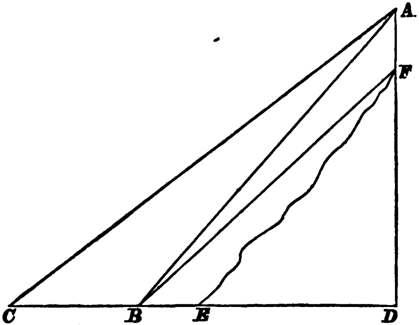
$$\therefore\ A\,B = a\,.\,\sin.\,\alpha\,.\,\text{cosec.}\,(\beta - \alpha).$$

$$y = A\,B\,.\,\sin.\,\beta\,. = a\,.\,\sin.\,\alpha\,.\,\sin.\,\beta\,.\,\text{cosec.}\,(\beta - \alpha).$$

$$x = A\,B\,.\,\cos.\,\beta\,. = a\,.\,\sin.\,\alpha\,.\,\cos.\,\beta\,.\,\text{cosec.}\,(\beta - \alpha).$$

(2) Suppose that the object $A\,F$, whose height is required stands on a hill $E\,F$.

FIG. X.



By the last example,

$$A\,D = a \,.\, \sin.\, \alpha \,.\, \sin.\, \beta \,.\, \operatorname{cosec.}\, (\beta - \alpha).$$
$$B\,D = a \,.\, \sin.\, \alpha \,.\, \cos.\, \beta \,.\, \operatorname{cosec.}\, (\beta - \alpha).$$

Let the angle $F\,B\,D$ be observed $= \gamma$,

$$\text{then } \frac{F\,D}{B\,D} = \tan.\, \gamma.$$
$$\therefore F\,D = B\,D \,.\, \tan.\, \gamma$$
$$= a \,.\, \sin.\, \alpha \,.\, \cos.\, \beta \,.\, \tan.\, \gamma \,.\, \operatorname{cosec.}\, (\beta - \alpha.)$$

$A\,D$ and $F\,D$ being thus determined, $A\,F$ is known; being $= A\,D - F\,D$.

Example. Suppose $\alpha = 20^\circ\ 35'$

$\beta = 25^\circ\ 42'$

$\gamma = 21^\circ\ 53'$

$a = 300$ yards.

Then $A\,D = a \,.\, \sin.\, \alpha \,.\, \sin.\, \beta \,.\, \operatorname{cosec.}\, (\beta - \alpha)$.

$$\begin{aligned} \log.\, a &= 2\cdot4771213 \\ \log.\, \sin.\, \alpha &= \bar{1}\cdot5460110 \\ \log.\, \sin.\, \beta \,. &= \bar{1}\cdot6371484 \\ \log.\, \operatorname{cosec.}\, (\beta - \alpha) &= 1\cdot0497129 \\ \hline \log.\text{ of } 512\cdot8538 &= 2\cdot7099936 \end{aligned}$$

$$\therefore A\,D = 512\cdot8538.$$

Again, $F\,D = a \,.\, \sin.\, \alpha \,.\, \cos.\, \beta \,.\, \tan.\, \gamma \,.\, \operatorname{cosec.}\, (\beta - \alpha)$.

$$\begin{aligned} \log.\, a &= 2\cdot4771213 \\ \log.\, \sin.\, \alpha &= \bar{1}\cdot5460110 \\ \log.\, \cos.\, \beta &= \bar{1}\cdot9547619 \\ \log.\, \tan.\, \gamma &= \bar{1}\cdot6038581 \\ \log.\, \operatorname{cosec.}\, (\beta - \alpha) &= 1\cdot0497129 \\ \hline \log.\text{ of } 428\cdot0211 &= 2\cdot6314652 \end{aligned}$$

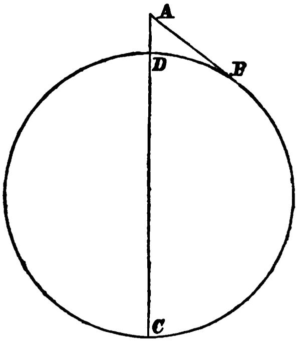
$$\therefore F\,D = 428\cdot0211$$

and $A\,F = A\,D - F\,D = 84\cdot8327$ yards.

259. If h = the height of the eye in feet above the level of the sea, d the distance in miles of the visible horizon,

$$d = \sqrt{\frac{3}{2}h} \text{ very nearly.}$$

Fig. XI.



$$\text{Let } A\,D = h \text{ feet} = \frac{h}{5280} \text{ miles,}$$

$$A\,B = d \text{ miles,}$$

$$C\,D = 2\,r = 7920 \text{ miles.}$$

Now $C\,A\,.\,A\,D = A\,B^2$. Euc. III. 36.,

$$\text{or } \left(2\,r + \frac{h}{5280}\right)\frac{h}{5280} = d^2$$

$$\frac{2\,r\,h}{5280} + \left(\frac{h}{5280}\right)^2 = d^2$$

$$\frac{7920\,h}{5280} + \left(\frac{h}{5280}\right)^2 = d^2$$

Rejecting $\left(\frac{h}{5280}\right)^2$ as very small,

$$\text{we have } \frac{3}{2}\,h = d^2$$

$$\text{and } d = \sqrt{\frac{3}{2}\,h}$$

Hence if to the height of the eye in feet above the level of the sea we add its half and take the square root, we shall have the distance of the visible horizon in miles.

Ex. Suppose the height of the eye above the level of the sea to be 24 feet, what will be the distance of the visible horizon?

$$\sqrt{24 + 12} = \sqrt{36} = 6$$

$\therefore$ the distance is 6 miles.

If both the eye and the object are above the level of the sea, then the distance of the object from the eye is equal to the sum of the distances answering to the two heights.

FIG. XII.

Ex. A lighthouse is 216 feet above the level of the sea, at what distance can it be seen by an eye 34 feet above the water?

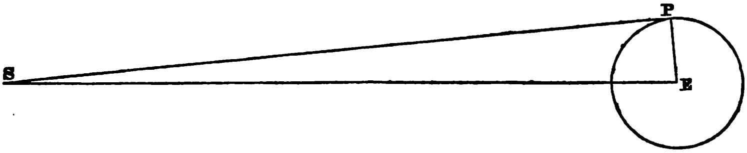
$$\sqrt{216 + \tfrac{1}{2}(216)} = \sqrt{324} = 18 \text{ miles,}$$

$$\text{and } \sqrt{54 + \tfrac{1}{2}(54)} = \sqrt{81} = 9 \text{ ditto.}$$

$\therefore$ the light can be seen at a distance of 27 miles.

260. The angle which the earth's radius subtends at the sun being $8''\cdot5776$, find the distance of the sun from the earth in terms of the earth's radius.

FIG. XIII.



Let S represent the sun.

E „ the earth.

Now $S\,P\,E$ is a right angle,

$$\therefore \frac{P\,E}{S\,E} = \sin.\ P\,S\,E = \sin.\ 8''\cdot5776.$$

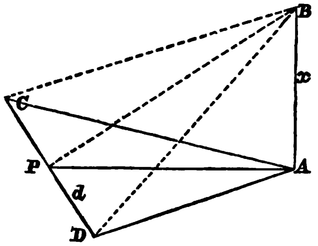
$$\text{Whence } S\,E = \frac{P\,E}{\sin.\ P\,S\,E} = \frac{\text{earth's radius}}{\sin.\ 8''\cdot5776}$$

$$= \frac{\text{earth's radius}}{8''\cdot5776 \times \sin.\ 1''.}$$

$$= 24046\cdot91 \times \text{earth's radius.}$$

261. In September 1864, a base, $D\,C$, being measured on Blackheath, of a mile in length, and the angles of elevation of Mr. Glaisher's balloon being taken, at the same time, by observers placed at its two extremities and in the middle; the angle at C being 54° 30′; that at P, 55° 8′; and that at D, 46° 10′: required the height, $B\,A$, of the balloon.

FIG. XIV.



Let $AB = x$; $DP = d$;

$\angle BCA = \alpha$; $BPA = \beta$: $BDA = \gamma$

$$\text{Then } AD = x\,.\,\text{cot.}\,\gamma.$$

$$AP = x\,\text{cot.}\,\beta.$$

$$AC = x\,\text{cot.}\,\alpha.$$

$$\text{Also cos. } APC\ (235) = \frac{CP^2 + AP^2 - AC^2}{2\,CP\,.\,AP};$$

$$\text{cos. } APD = -\,\text{cos. } APC = \frac{AP^2 + DP^2 - AD^2}{2\,AP\,.\,DP.}.$$

$$\therefore \frac{CP^2 + AP^2 - AC^2}{CP} = -\,\frac{AP^2 + DP^2 - AD^2}{PD}.$$

Substituting and transposing,

$$PD(d^2 + x^2.\text{cot.}^2\beta - x^2.\text{cot.}^2 d) = -\,CP\,(x^2.\text{cot.}^2\beta + d^2 - x^2.\text{cot.}^2\gamma).$$

$$\therefore x^2.\ \text{cot.}^2\,\gamma + x^2.\ \text{cot.}^2\,\alpha - 2\,x^2.\ \text{cot.}^2\,\beta = 2\,d^2.$$

$$x^2\,(\text{cot.}^2\,\gamma + \text{cot.}^2\,\alpha - 2\ \text{cot.}^2\,\beta) = 2\,d^2$$

$$x = \frac{d\,.\sqrt{2}}{\sqrt{\text{cot.}^2\,\gamma + \text{cot.}^2\,\alpha - 2\ \text{cot.}^2\,\beta}}$$

$$\begin{array}{rl}
d = 880 \text{ yards} & \\
\text{cot.}^2\,\gamma = \text{cot.}^2\,46^\circ\,10' = & \cdot 9217592 \\
\text{cot.}^2\,\alpha = \text{cot.}^2\,54^\circ\,30' = & \cdot 5087870 \\
\hline
 & 1\cdot 4305462 \\
2\ \text{cot.}^2\,\beta = 2\ \text{cot.}^2\,55^\circ\,8' = & \cdot 9709074 \\
\hline
 & \cdot 4596388
\end{array}$$

$$\therefore x = \frac{880\sqrt{2}}{\sqrt{0\cdot 4596388}} = \frac{880}{\sqrt{\cdot 2298194}}$$

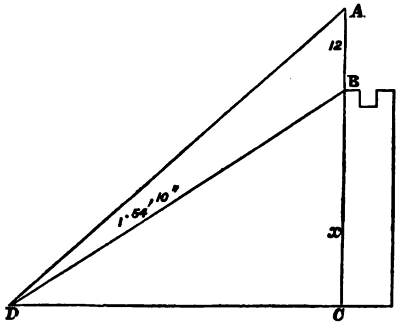
$$= 1835\cdot 648 \text{ yards.}$$

NOTE.—If the observations were made at unequal distances; or, if CP and PD were respectively d and e, the expression would become:

$$x = \sqrt{\frac{(d + e)\,d\,e}{d\,.\,\text{cot.}^2\,\gamma + e\,.\,\text{cot.}^2\,\alpha - (d + e)\ \text{cot.}^2\,\beta}}.$$

262. An object 12 feet high standing on the top of a tower, subtends an angle 1° 54′ 10″ at a station which is 250 feet from the base of the tower; find the height of the tower.

FIG. XV.



Let $B\,C =$ height of tower $= x$.
$A\,B =$,, object above $= 12$ feet.
$\angle A\,D\,C = \angle A$
$\angle B\,D\,C = \angle B$
By hyp. $D\,C = 250$ feet and $\angle (A - B) = 1°.\ 54'.\ 10''$.

$$\text{Now tan. } A = \frac{x + 12}{250},\ \tan B = \frac{x}{250};$$

$$\therefore \tan.(A - B) = \frac{\dfrac{x + 12}{250} - \dfrac{x}{250}}{1 + \dfrac{x.(x + 12)}{250^2}} = \tan.\ 1°.\ 54'.\ 10''.$$

∴ taking from tables tan. 1°. 54′. 10″ = ·033222, and multiplying both terms of the fraction by 250^2,

$$\frac{12 \times 250}{62500 + x^2 + 12\,x} = \cdot 033222.$$

$$\text{Whence } x^2 + 12\,x = 27801\cdot 6073.$$

$$\therefore x = -6 \pm \sqrt{27837\cdot 6073}$$
$$= -6 + 166\cdot 846$$
$$= 160\cdot 846 \text{ feet, the height required.}$$

CHAPTER XII.

CONIC SECTIONS REFERRED TO RECTANGULAR CO-ORDINATES.

⁂ Conic sections, as the words indicate, are the curves formed by the intersection of a plane with a cone. In their widest application, they describe the course of the planets, and the movement of projectiles through space generally.

A plane *may* meet a cone in a point, in a straight line, or a circle, but the curves which are more exclusively considered conic sections, are the ellipse, the parabola, and the hyperbola.

ON THE POSITION OF A POINT.

263. The position of a point, in a plane, is determined by referring it to two straight lines, which intersect each other at a given angle, in the same plane.

These straight lines are called rectangular or oblique co-ordinate axes, according as the angle at which they intersect, is, or is not, a right angle.

The point in which they intersect is called the origin of co-ordinates.

Let A (Fig. I.) be the origin of co-ordinates, $X A X'$, $Y A Y'$ the rectangular co-ordinate axes, P any point and $P M$, $P N$ the perpendiculars let fall from it upon the co-ordinate axes, these perpendiculars are called the rectangular co-ordinates of P.

Instead of the perpendicular $M P$, its equal, $A N$, is commonly used to determine the position of the point P.

For the sake of distinctness $A N$ is called the *abscissa* and $N P$ the *ordinate.*

The line $A X$ along which the abscissæ are measured is called the axis of x, and the line $A Y$ in the direction of which the ordinates are measured, the axis of y.

The position of the point P will be determined if we know the values of its two co-ordinates.

When the co-ordinates are unknown, $A N$ is denoted by x, and $N P$ by y. When they are supposed to be known they are represented by the first letters of the alphabet, a, b, &c., or by the accented letters x', y', or x'', y'', and the point is called the point $(a, b,)$ the point $(x', y',)$ &c.

The convention pointed out in Art. 216 is also adopted here.

If the point (x', y') be in the angle

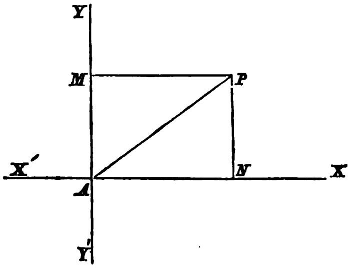
$Y A X$, both x' and y' are positive;
if in $Y A X'$, x' is negative and y' positive;
„ $X' A Y'$, both x' and y' are negative;
„ $Y' A X$, x' is positive and y' negative.

When the point is situated *upon* the axis $A X$, then x' is positive and $y' = 0$; and when upon the axis $A Y$, then $x' = 0$ and y' is positive.

For the origin $x' = 0$ and $y = 0$.

264. *To find the distance of a point from the origin in terms of its co-ordinates.*

FIG. I.



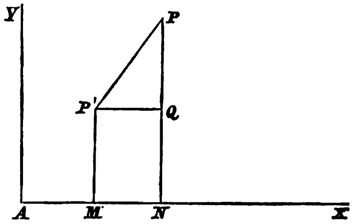
Let P be the given point, $A N = x'$; $N P = y'$, its given co-ordinates. Join $A P$, and let $A P = d$;

$$\text{then } A P^2 = A N^2 + N P^2, \text{ or } d^2 = x'^2 + y'^2,$$

$$\therefore\ d = \sqrt{x'^2 + y'^2}.$$

265. *To find the distance between two points in terms of their co-ordinates.*

FIG. II.



Let P' be a point whose co-ordinates are x' and y', and P any other point whose co-ordinates are x and y.

Join $P'P$ and draw $P'Q$ parallel to AX, and meeting the ordinate of P in Q, and let $P'P = d$.

$$\text{Then } P'P^2 = P'Q^2 + PQ^2 \text{ or } d^2 = (x - x')^2 + (y - y')^2$$

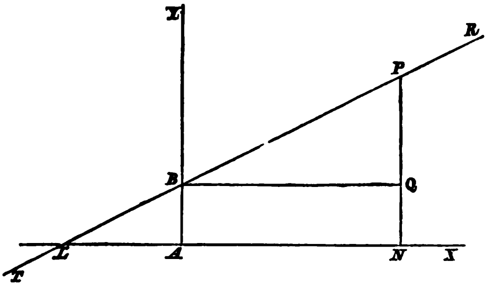
$$\therefore\ d = \sqrt{(x - x')^2 + (y - y')^2}.$$

ON THE STRAIGHT LINE.

⁂ The equation to a straight line expresses the relation between the co-ordinates of any point in the line.

266. *To find the equation to a straight line.*

FIG. III.



Let A be the origin, AX, AY the axes of x and y respectively, RT the straight line whose equation it is proposed to find, P any point in it, $AN = x$; $NP = y$; the co-ordinates of the point P, $AB = b$; and tan. $RLX = a$.

Draw BQ parallel to AX, meeting NP in Q.

$$\text{Then } \frac{PQ}{BQ} = \text{tan. } PBQ = \text{tan. } RLX,$$

$$\text{or } \frac{y - b}{x} = a.$$

Whence $y = ax + b$, the equation required . . . . (I.)

In this equation a is the tangent of the angle which that part of the line which falls above the axis of x, makes with the axis of x produced. b is the distance between the origin and the point where the given line cuts the axis of y. a and b are the same for the same line, but differ for different lines and are called *arbitrary constants*.

If $b = 0$ the line passes through the origin, and its equation is $y = ax$.

In the equation $y = ax + b$,

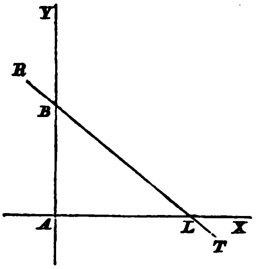
$$a = \text{tan. } RLX = \frac{AB}{AL} = \frac{b}{c} \text{ if } AL = c.$$

$$\therefore\ y = \frac{b}{c}x + b. \qquad \text{. . . . . (II.)}$$

and $\frac{y}{b} - \frac{x}{c} = 1$, another form of the equation, where c, b are the distances from the origin where the line cuts the axis of x and y respectively.

267. If the straight line cuts the positive parts of both the axes as $R\,T$, then

FIG. IV.



$$a = \tan.\, R\,L\,X = -\tan.\, R\,L\,A = -\frac{A\,B}{A\,L} = -\frac{b}{c},$$

and the equation becomes $\frac{y}{b} + \frac{x}{c} = 1$ . . . . (III.)

Of these three forms, I. is most frequently referred to.

268. The *locus* of the indeterminate equation of the first degree $A\,y + B\,x + C = 0$, is a straight line; for dividing by A and transposing, we obtain

$$y = -\frac{B}{A}x - \frac{C}{A},$$

and denoting $-\frac{B}{A}$ by a, and $-\frac{C}{A}$ by b, we have $y = a\,x + b$, coinciding with the equation to a straight line just found.

269. *To construct the line whose equation is* $y = a\,x + b$.

It will be sufficient to find two points of the proposed line, since two points only are necessary to fix the position of a straight line. The two points most readily found are those in which the line cuts the axis of x and y, and which are determined by making x and y successively $= 0$ in the given equation.

Ex. Construct the line whose equation is

$$y = -a\,x + b.$$

$$\text{Let } x = 0 \;\therefore\; y = b.$$

In $A\,Y$, fig. IV., take $A\,B = b$, the value of y when $x = 0$.

$$\text{Let } y = 0 \;\therefore\; x = \frac{b}{a}.$$

In $A\,X$ take $A\,L = \frac{b}{a}$, join B, L, by a straight line and produce it both ways, then the indefinite line $R\,T$ is that required.

270. *To find the equation to a straight line which passes through a given point* $(x', y'.)$

The general equation to a straight line is

$$y = a\,x + b,$$

and (x', y') being a point in the line we must have

$$y' = a\,x' + b.$$

Subtracting this equation from the former we obtain

$$y - y' = a\,(x - x') \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (\alpha.)$$

the equation required.

The co-efficient a is indeterminate because an infinite number of lines may be drawn all passing through the same point.

If the line passes through *two* given points its position is fixed, and a, which is the tangent of the angle it makes with the axis of x, must admit of being expressed in terms of the given co-ordinates.

Let the two given points be (x', y'), (x'', y''),

$$\therefore y' = a\,x' + b,$$

$$y'' = a\,x'' + b,$$

$$\text{whence } a = \frac{y'' - y'}{x'' - x'}.$$

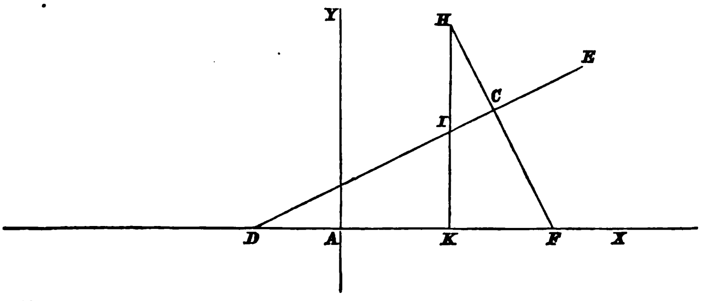
Substituting this value of a in (α) we have

$$y - y' = \frac{y'' - y'}{x'' - x'}\,(x - x') \;.\;.\;.\;.\; (\beta)$$

the equation to a straight line passing through two given points.

271. *To draw through a given point a straight line perpendicular to a given straight line.*

FIG. V.



Let H be the given point (x', y'), $D\,E$ the given line, $y = a\,x + b$ its equation.

Through H draw $H\,F$ perpendicular to $D\,E$, cutting $D\,E$ in C. Then since $H\,F$ passes through a given point, its equation is of the form,

$$y - y' = a(x - x'),$$

where $a = \tan. HFX = -\tan. HFD,$

$= -\cot. CDF \because CDF$ is the complement of $CFD,$

$= -\dfrac{1}{\tan. CDF} = -\dfrac{1}{a}$; substituting, we have

$y - y' = -\dfrac{1}{a}(x - x')$, the equation required.

Hence it appears that the equation to a straight line, perpendicular to the straight line whose equation is

$$y = ax + b,$$

is $y = -\dfrac{1}{a}x + b'$.

To find the length of the perpendicular HC, we have

$HK = y'$ and $IK = ax' + b;$

$\therefore HI = y' - ax' - b,$

but $HC = HI . \cos. IHC = HI . \cos. CDF$

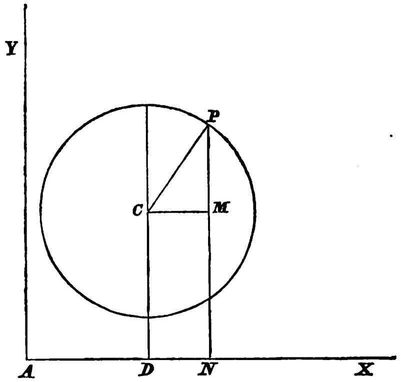
$$= \frac{HI}{\sec. CDF} = \frac{HI}{\sqrt{1 + \tan.^2 CDF}} = \frac{y' - ax' - b}{\sqrt{a^2 + 1}}.$$

ON THE CIRCLE.

⁂ If the relation between the co-ordinates of every point of a curve be expressed by an equation, the curve is called the *locus* of the equation.

272. *To find the equation to the circle.*

FIG. VI.



Let C be the centre of the circle whose equation it is proposed to find. Assume the point A without the circle as the origin, and let P be any point in the circle,

$$A\,N = x;\ N\,P = y;\ \text{and the radius } C\,P = r.$$

The co-ordinates of the centre are $A\,D = x'$; $D\,C = y'$;

$$\text{Now } C\,M^2 + M\,P^2 = C\,P^2,$$

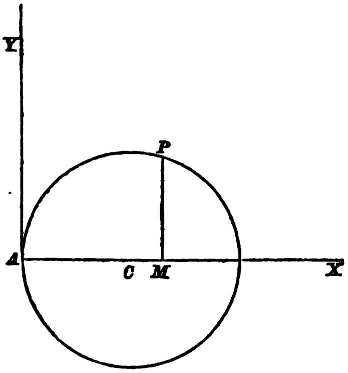
$$\text{or } (A\,N - A\,D)^2 + (N\,P - C\,D)^2 = C\,P^2,$$

$$\therefore\ (x - x')^2 + (y - y')^2 = r^2 \ \ldots\ldots\ldots \text{(I.)}$$

the equation required.

273. Let the origin be at the extremity of any diameter.

FIG. VII.



In this case we have for the co-ordinates of the centre $x' = r$, $y' = 0$, and equation (I.) becomes

$$y^2 = 2\,r\,x - x^2 \ldots\ldots\ldots\ldots \text{(II.)}$$

Let the origin be at the centre, then $x' = 0, y' = 0$, and we have

$$x^2 + y^2 = r^2 \ldots\ldots\ldots\ldots \text{(III.)}$$

Equation (I.), when developed, becomes

$$y^2 + x^2 - 2\,y\,y' - 2\,x\,x' + y'^2 + x'^2 - r^2 = 0,$$

but the general equation of the second degree between two variables is

$$a\,y^2 + b\,x\,y + c\,x^2 + A\,y + B\,x + C = 0.$$

Comparing the corresponding terms of these two equations, we have

$$a = 1;\ b = 0;\ c = 1;$$

whence it appears that the general form of the equation to the circle, when referred to rectangular co-ordinates, is

$$y^2 + x^2 + A\,y + B\,x + C = 0.$$

274. *To find the equation to the tangent to a circle.*

The following is the method commonly used:—

Take a point in the circle whose co-ordinates are x', y', and a contiguous point (x'', y'').

The equation to a straight line passing through two points (x', y'), (x'', y'') is by 270 (β).

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad \ldots\ldots\ldots\ldots \quad (\alpha)$$

but since these points are in the circle, we must have

$$x'^2 + y'^2 = r^2 \quad \ldots\ldots\ldots\ldots \quad (\beta)$$
$$x''^2 + y''^2 = r^2 \quad \ldots\ldots\ldots\ldots \quad (\gamma)$$

subtracting (β) from (γ) we obtain

$$(y''^2 - y'^2) + (x''^2 - x'^2) = 0,$$
$$\text{or } (y'' + y')(y'' - y') + (x'' + x')(x'' - x') = 0.$$
$$\therefore \frac{y'' - y'}{x'' - x'} = -\frac{x'' + x'}{y'' + y'}.$$

Substituting in (a) we have

$$y - y' = -\frac{x'' + x'}{y'' + y'}(x - x').$$

This line is a secant; but if we suppose the points to approach indefinitely near each other, ultimately it becomes a tangent.

Hence, making $x'' = x'$, $y'' = y'$, we obtain

$$y - y' = -\frac{x'}{y'}(x - x') \text{ the equation required.}$$

Multiplying both sides by y' and transposing we have,

$$y\,y' + x\,x' = y'^2 + x'^2;$$

or $y\,y' + x\,x' = r^2$, a simpler form of the equation to the tangent.

275. If a straight line be drawn from the point of contact at right angles to the tangent, it is called the *normal.*

Since $y - y' = -\frac{x'}{y'}(x - x')$ is the equation to the tangent,

$\therefore$ by (271) the equation to the normal will be $y - y' = \frac{y'}{x'}(x - x')$.

Multiplying by x', we have $y\,x' = x\,y'$;

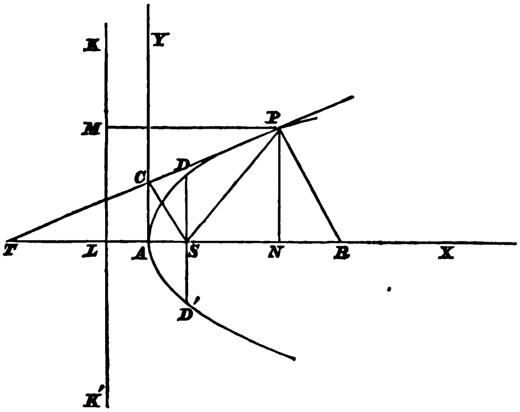
whence $y = \frac{y'}{x'}\,x$.

ON THE PARABOLA.

⁂ The parabola is the locus of a point whose distance from a given point is always equal to its perpendicular distance from a given fixed line.

276. *To find the equation to the parabola.*

FIG. VIII.



Let S be the given point or focus, $K\,K'$ the given straight line or directrix, then if P be taken such that $P\,S$ always equals $P\,M$, the locus of P is the parabola.

Through S draw $L\,S$ perpendicular to $K\,K'$ and bisect it in A, then A is a point in the curve.

Since $A\,S$ is known let $A\,S = a$.

Assume $A\,X$, $A\,Y$ as the rectangular axes, and let A or the vertex be the origin, $A\,N = x$; $N\,P = y$.

$$\text{Now } S\,P^2 = P\,M^2, \text{ or } S\,N^2 + P\,N^2 = L\,N^2,$$

$$\text{or } (x - a)^2 + y^2 = (a + x)^2,$$

$$\therefore y^2 = 4\,a\,x, \text{ the equation required.}$$

277. *To trace the figure of the parabola from its equation.*

$$\text{Since } y^2 = 4\,a\,x \;\therefore y = \pm\, 2\sqrt{a\,x},$$

let $x = 0 \;\therefore y = 0$, and the curve passes through the origin.

Let x have any positive value, then, for every value of x there will be two values of y with contrary signs; therefore as x increases from zero to infinity, y also increases from 0 to $\pm \infty$, so that the parabola has two infinite branches symmetrically situated with respect to the axis $A\,X$.

278. Let x have any negative value, then y being in this case imaginary, no part of the curve can lie to the left of A, therefore the parabola can only have one focus and one directrix.

279. The double ordinate $B\,D$ passing through S is called the parameter or *latus rectum.* The same term is used for the ordinate drawn through the focus of the ellipse or hyperbola.

To find the value of the latus rectum, we have at the point S, $x = a$ $\therefore$ $y^2 = 4a^2$ and $y = 2a = SB$ $\therefore$ $BD = 4a$.

If $4a = p$, then the equation to the parabola becomes

$$y^2 = px.$$

279*. *To express the distance of any point in the parabola from the focus, in terms of its abscissa.*

$$\text{By definition } SP = PM$$
$$= NL = NA + AL.$$
$$\therefore SP = x + a.$$

280. *To find the equation to the tangent to the parabola.*

The equation to a straight line passing through the points (x', y'), (x'', y''), is (270 β).

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x') \quad \ldots\ldots\ldots\ldots (\alpha)$$

but these points being in the parabola, we have

$$y'^2 = 4ax' \quad \ldots\ldots\ldots\ldots (\beta)$$
$$y''^2 = 4ax'' \quad \ldots\ldots\ldots\ldots (\gamma)$$

Subtracting (β) from (γ), $y''^2 - y'^2 = 4a(x'' - x')$,

$$\text{or } (y'' + y')(y'' - y') = 4a(x'' - x').$$
$$\therefore \frac{y'' - y'}{x'' - x'} = \frac{4a}{y'' + y'},$$

$\therefore$ (α) becomes by substitution

$$y - y' = \frac{4a}{y'' + y'}(x - x').$$

Now let the point (x'', y'') be supposed to coincide with (x', y'), then $x'' = x'$, $y'' = y'$.

$\therefore$ $y - y' = \dfrac{2a}{y'}(x - x')$ the equation to the tangent required.

Again, multiplying both sides by y', we have

$$y y' - y'^2 = 2a(x - x'),$$
$$\text{but } y'^2 = 4ax';$$
$$\therefore y y' = 2ax - 2ax' + 4ax'$$
$$= 2a(x + x') \text{ the equation most commonly used.}$$

281. If PR be drawn perpendicular to the tangent PT from the point P, it is called the *normal*, NT is the *subtangent*, NR *the subnormal.*

282. *To find the equation to the normal.*

The equation to the tangent is

$$y - y' = \frac{2a}{y'}(x - x');$$

but since $P R$ is perpendicular to $P T$,

$\therefore$ by (271) the equation to the normal will be

$$y - y' = -\frac{y'}{2a}(x - x').$$

283. *To find the value of the subtangent.*

If in the equation $y y' = 2a(x + x')$ we make $y = 0$,

we have $x = -x'$, that is, $A N = A T$,

or the subtangent is equal to twice the abscissa, a very useful and important property.

284. *To find the value of the subnormal.*

In the equation to the normal,

$$y - y' = -\frac{y'}{2a}(x - x'),$$

put $y = 0 \therefore x - x' = 2a = A R - A N = N R$, a *constant.*

The length of the tangent is $\sqrt{N T^2 + N P^2}$,

$$= \sqrt{4 x'^2 + y'^2} = 2\sqrt{x'^2 + a x'}.$$

The length of the normal is $\sqrt{N R^2 + N P^2}$,

$$= \sqrt{4 a^2 + y'^2} = 2\sqrt{a^2 + a x'}.$$

285. *The locus of the foot of the perpendicular dropped from the focus upon the tangent to a parabola is in the line touching the parabola at its vertex.*

Let $P T$, the tangent at P, meet $A Y$ in C, and join $S C$.

Because the subtangent is equal to twice the abscissa we have $T A = A N = x$;

$\therefore$ by similar triangles $T C = C P$.

Also $T S = x + a = S P$,

and the triangles $S P C$, $S T C$ are equal;

$\therefore$ $S C$ is perpendicular to $P T$.

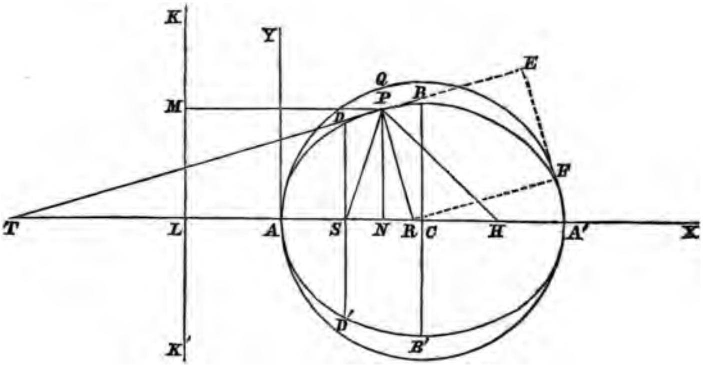
Hence the tangent at any point, and the perpendicular upon it from the focus, intersect in the line which touches the parabola at the vertex.

ON THE ELLIPSE.

⁂ The ellipse is the locus of a point whose distance from a given point is always less in a given ratio than its distance from a given fixed line.

286. *To find the equation to the ellipse.*

FIG. IX.



Let S be the given point or focus, $K\,K'$ the given line or directrix, P any point in the ellipse. Through S draw the indefinite line $L\,S\,X$ perpendicular to $K\,K'$, join $S\,P$, and draw $P\,M$, $P\,N$, perpendicular to $K\,K'$ and $L\,X$ respectively.

Let the given ratio of $S\,P : P\,M$ be $e : 1$, e being < 1.

Divide $S\,L$ in A so that $S\,A : A\,L :: e : 1$,
then A is a point in the ellipse.

Since $A\,S$ is known, let it $= m$ $\therefore A\,L = \frac{m}{e}$, and from what precedes $S\,P = e\,.\,P\,M$.

From A draw $A\,Y$ perpendicular to $A\,X$, then $A\,X$, $A\,Y$, will be the co-ordinate axes to which the figure is to be referred, and $A\,N = x$; $N\,P = y$;

$$\text{then } S\,P^2 = e^2\,.\,P\,M^2 = e^2\,(A\,L + A\,N)^2,$$

$$\text{or } S\,N^2 + N\,P^2 = e^2\left(\frac{m}{e} + x\right)^2,$$

$$\text{or } (x - m)^2 + y^2 = (m + e\,x)^2,$$

$$\text{whence } y^2 = 2\,m\,(1 + e)\,x - (1 - e^2)\,x^2.$$

Multiply and divide both terms of the right member of this equation by $(1 - e^2)$ which will not alter its value;

$$\text{then } y^2 = (1 - e^2)\left\{\frac{2\,m\,(1 + e)\,x}{1 - e^2} - \frac{(1 - e^2)}{1 - e^2}\,x^2\right\}$$

$$= (1 - e^2)\left(\frac{2\,m}{1 - e}\,x - x^2\right).$$

$$\text{Since } \frac{m}{1 - e} \text{ is known, let it} = a,$$

$$\therefore\ y^2 = (1 - e^2)\,(2\,a\,x - x^2)$$

the equation required.

287. *To determine the points where the curve cuts the axis of x.*

In the equation $y^2 = (1 - e^2)(2\,a\,x - x^2)$

$$\text{let } y = o \therefore x = o, \text{ or } x = 2\,a;$$

$$x = o, \text{ gives the point } A.$$

$$x = 2\,a, \text{ gives the point } A'.$$

288. *To determine the points where the curve cuts the axis $B\,B'$.*

Bisect $A\,A'$ in C, then at this point $x = a$, and we have

$$y^2 = a^2(1 - e^2),$$

whence $y = \pm\, a\sqrt{1 - e^2}$, which is always real since $e < 1$. Hence if $B\,B'$ be drawn through C perpendicular to $A\,A'$; and $C\,B$, $C\,B'$ be each taken equal to $a\sqrt{1 - e^2}$, the points $B\,B'$ will be in the ellipse. If we denote $B\,C\,B'$ by $2\,b$, then $b = \pm\, a\sqrt{1 - e^2}$, and by squaring and transposing, $1 - e^2 = \frac{b^2}{a^2}$.

Substituting for $(1 - e^2)$ in the equation to the ellipse, we have

$$y^2 = \frac{b^2}{a^2}(2\,a\,x - x^2) \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \; \text{(I.)}$$

$A\,A'$ is called the major axis.
$B\,B'$ „ „ minor „

A and A' are the vertices and C the centre of the ellipse.

289. If we make the centre the origin of co-ordinates, the axis major becomes the co-ordinate axis of x, and the axis minor the co-ordinate axis of y; we must then change x into $a + x'$, and

$$y^2 = \frac{b^2}{a^2}\left\{2\,a\,(a + x') - (a + x')^2\right\}$$

$$= \frac{b^2}{a^2}(a^2 - x'^2) \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \; \text{(II.)}$$

Multiplying both sides by a^2 and transposing, also suppressing the accent which was used only to distinguish the new from the old abscissa, we have

$$a^2\,y^2 + b^2\,x^2 = a^2\,b^2 \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \; \text{(III.)}$$

Dividing each term by $a^2\,b^2$ we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \; \text{(IV.)}$$

290. If with the centre for the origin we make the axis minor the axis of x, the axis major will become the axis of y. x and y interchanging, the equation becomes

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \;. \; \text{(V.)}$$

If we take the axis minor and the tangent at its extremity as axes, x in the last equation must be changed into $x - b$; then we have

$$\frac{(x-b)^2}{b^2} + \frac{y^2}{a^2} = 1,$$

$$\text{or reducing } y^2 = \frac{a^2}{b}(2\,b\,x - x^2) \;\ldots\ldots\ldots\ldots \text{(VI.)}$$

291. *To trace the figure of the ellipse from its equation.*

Resuming (II.), we have $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$.

Let $x = 0 \quad \therefore y = \pm b = C\,B$ or $C\,B'$.

Suppose x to have any positive or negative value less than a; then for every such value of x there will be two values of y with contrary signs.

Let $x = \pm a \quad \therefore y = 0$; that is, the curve cuts the axis in the points A and A'.

Let $x > \pm a$. In this case the values of y are imaginary; therefore no part of the curve lies beyond A to the left, or A' to the right.

It appears, therefore, that the ellipse is a continuous curve returning into itself, and divided by the axis major into two equal parts.

292. Since $\frac{m}{1-e}$ or $\frac{A\,S}{1-e} = a \quad \therefore m$ or $A\,S = a\,(1 - e)$,

also $A\,S : A\,L :: e : 1 \quad \therefore \frac{m}{e} = A\,L = \frac{a\,(1-e)}{e}$.

The distance between the centre and the focus is $C\,S, = A\,C - A\,S = a - a\,(1 - e) = a\,e$; and $\frac{C\,S}{C\,A} = e$, which quantity is called the eccentricity.

293. *To find the value of the ordinate passing through the focus.*

In this case $x = a\,e$,

$$\begin{aligned} \therefore y^2 &= \frac{b^2}{a^2}(a^2 - a^2 e^2) \\ &= b^2(1 - e^2) \\ &= \frac{b^4}{a^2} \quad \because (1 - e^2) = \frac{b^2}{a^2}; \\ \therefore y &= \frac{b^2}{a}. \end{aligned}$$

294. *To express the distance of any point in the ellipse from the focus in terms of the abscissa.*

By definition $S\,P = e\,.\,P\,M = e\,(C\,L - C\,N)$,

but $C\,L = C\,A + A\,L = a + \frac{a\,(1-e)}{e} = \frac{a}{e}$,

$$\therefore S\,P = e\left(\frac{a}{e} - x\right) = a - e\,x.$$

In the same way, by drawing a new directrix on the right of A', it may be shown that $H P = a + e x$.

Adding, we get $S P + H P = 2 a =$ the major axis.

$H P$ and $S P$ are called the *focal distances* of the point P. Hence an easy mechanical method of describing an ellipse.

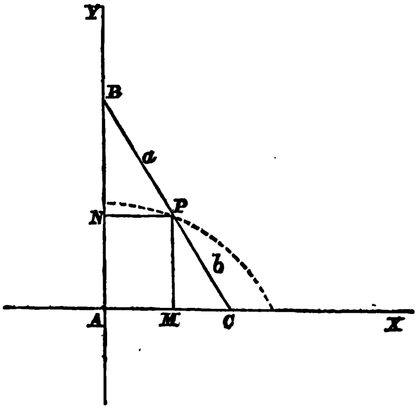
Fix two pins in the foci S, H, to which attach a thread (the less extensible the better) equal in length to the major axis. Then placing a pen or pencil so as to keep the string constantly stretched, let it be carried quite round, and it will trace the curve with considerable accuracy.

295. *To describe an ellipse of which the semi-axes are a and b.*

Take a straight line, $B C$, divided in any point P, so that $B P : P C :: a : b$; and let $A X$, $A Y$ be the axes of x and y.

Move the line, $B C$, between the axes $A Y$, $A X$, so that its extremities are always on those axes; the curve traced out by the point P, will be a quarter of the required ellipse. Proceed similarly in the other quadrants until the curve is completed.

FIG. X.



$$\text{For } N P \text{ or } A M : B P :: M C : P C,$$

$$\text{or, } x : a :: \sqrt{b^2 - y^2} : b,$$

$$\therefore b^2 x^2 = a^2 b^2 - a^2 y^2$$

$$a^2 y^2 + b^2 x^2 = a^2 b^2$$

the equation III. to the ellipse already found.

If a ladder be placed against a wall, and its foot drawn along the ground at right angles to the wall, the middle step will trace out a quadrant of a circle, and any other step a quarter of an ellipse.

296. *To find the equation to the tangent to the ellipse.*

The equation to the straight line passing through the points (x', y'), (x'', y'') is (270 β)

$$y - y' = \frac{y'' - y'}{x'' - x'}(x - x').$$

But since these points are in the ellipse, we have

$$y'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$$

$$y''^2 = \frac{b^2}{a^2}(a^2 - x''^2)$$

$$\therefore y''^2 - y'^2 = -\frac{b^2}{a^2}(x''^2 - x'^2)$$

$$\text{or } (y'' + y')(y'' - y') = -\frac{b^2}{a^2}(x'' + x')(x'' - x')$$

$$\text{whence } \frac{y'' - y'}{x'' - x'} = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}.$$

Substituting, we have

$$y - y' = -\frac{b^2}{a^2} \cdot \frac{x'' + x'}{y'' + y'}(x - x').$$

Let the point (x'', y'') coincide with (x', y'),

$$\therefore y - y' = -\frac{b^2}{a^2} \cdot \frac{x'}{y'}(x - x')$$

the equation required.

Multiplying both sides by $a^2 y'$ and transposing,

$$\therefore a^2 y y' + . b^2 x x' = a^2 y'^2 + b^2 x'^2;$$

or by substituting in III., Art. 289,

$$a^2 y y' + b^2 x x' = a^2 b^2,$$

the form most commonly employed.

The equation to the *normal* will be (271)

$$y - y' = \frac{a^2 y'}{b^2 x'}(x - x').$$

297. When the major axis is supposed to become indefinitely great, the ellipse passes into a parabola.

Let C be the centre, and S the focus of an ellipse whose equation is

$$y^2 = \frac{b^2}{a^2}(2 a x - x^2) = \frac{2 b^2}{a} x - \frac{b^2}{a^2} x^2.$$

$$\text{Let } m = A S = A C - S C = a - \sqrt{a^2 - b^2}$$

$$\therefore b^2 = 2 a m - m^2.$$

Substituting, we get

$$y^2 = \left(4 m - \frac{2 m^2}{a}\right) x - \left(\frac{2 m}{a} - \frac{m^2}{a^2}\right) x^2.$$

Now if a be supposed to vary, this will be the equation to a series of ellipses, in which the distance $A S$ is the same for all, but the axis major different for each: thus giving to a any particular value, we have a corresponding ellipse. Now let a be infinite, then, since all the other terms vanish, the equation becomes

$$y^2 = 4\,m\,x$$

the equation to a parabola.

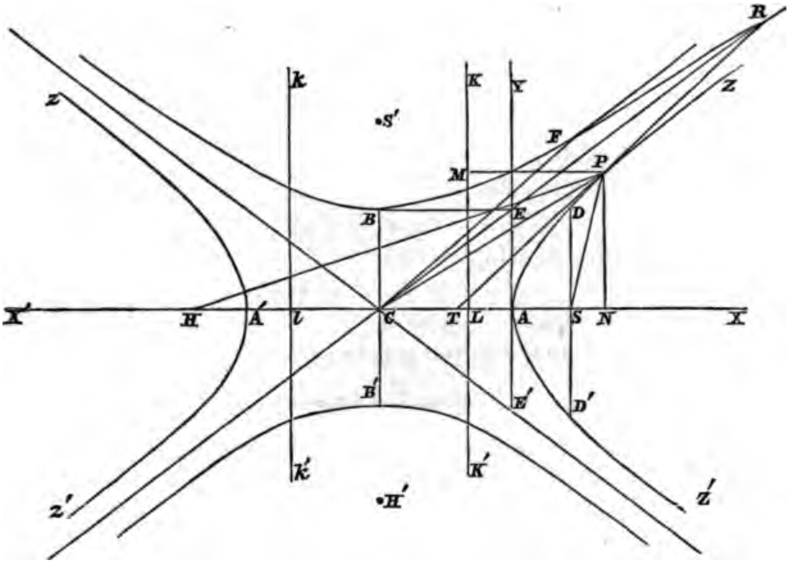
If x is very small when compared with a, the equation to the ellipse is very nearly that to a parabola; this is the reason why the path of a comet, when near its perihelion, appears to be a portion of a parabola.

ON THE HYPERBOLA.

⁂ The hyperbola is the locus of a point whose distance from a given point is always greater in a given ratio than its distance from a given fixed line.

298. *To find the equation to the hyperbola.*

FIG. XI.



Let S be the given point or focus, $K\,K'$ the given line or directrix, P any point in the hyperbola. Through S draw the indefinite line $L\,S\,X$ perpendicular to $K\,K'$. Join $S\,P$ and draw $P\,M$, $P\,N$ perpendicular to $K\,K'$ and $L\,X$ respectively.

Let the given ratio of $S\,P$ to $P\,M$ be $e : 1$, e being > 1.

Divide $S\,L$ in A so that $S\,A : A\,L :: e : 1$, then A is a point in the hyperbola.

Since $A\,S$ is known let it equal m,

$$\therefore\ A\,L = \frac{m}{e}, \text{ and } S\,P = e\,.\,P\,M.$$

From A draw AY perpendicular to AX, and let AX, AY be the co-ordinate axes, $AN = x$; $NP = y$;

$$\text{then } SP^2 = e^2 . PM^2 = e^2 (AL + AN)^2$$

$$\text{or } SN^2 + NP^2 = e^2 \left(\frac{m}{e} + x\right)^2$$

$$\text{or } (x - m)^2 + y^2 = (m + ex)^2$$

$$\text{whence } y^2 = 2m(1 + e)x + (e^2 - 1)x^2$$

$$= (e^2 - 1)\left(\frac{2mx}{e - 1} + x^2\right).$$

Let $\frac{m}{e - 1} = a$,

∴ $y^2 = (e^2 - 1)(2ax + x^2)$ the equation required.

299. *To determine the points where the curve cuts the axis of x.*

In the equation just found, let $y = 0$,

$$\therefore x = 0, \text{ or } = -2a;$$

$x = 0$ gives the point A, $x = -2a$ the point A'.

Bisect AA' in C, then at this point $x = -a$,

$$\text{and } y^2 = -a^2(e^2 - 1),$$

which is an impossible quantity, and thus the curve does not cut the axis of y.

If, however, BB' be drawn through C perpendicular to AA', and CB, CB', be each taken equal to $a\sqrt{e^2 - 1}$, we get the points B, B', which, though not in the hyperbola, are usually denoted by $2b$, and found as follows,—

$$b = \pm a\sqrt{e^2 - 1}, \text{ and } e^2 - 1 = \frac{b^2}{a^2};$$

substituting for $e^2 - 1$ in the equation to the hyperbola, we have,

$$y^2 = \frac{b^2}{a^2}(2ax + x^2) \text{ . . . . (I.)}$$

AA is the transverse or major axis, BB' the conjugate or minor axis, although it does not meet the curve. A and A' are the vertices, and C the centre of the hyperbola.

300. If we make the centre the origin, we must change x into $x' - a$, since $AN = CN - CA$;

$$\text{then } y^2 = \frac{b^2}{a^2}\left\{2a(x' - a) + (x' - a)^2\right\} = \frac{b^2}{a^2}(x'^2 - a^2) \text{ . (II.)}$$

301. Multiplying by a^2, transposing and suppressing the accent which was used only to distinguish the new from the old abscissa, we have,

$$a^2y^2 - b^2x^2 = -a^2b^2 \text{ . . . . (III.)}$$

302. Dividing each term by a^2b^2,

$$\therefore \frac{y^2}{b^2} - \frac{x^2}{a^2} = -1 \text{ . . . . (IV.)}$$

303. *To trace the figure of the hyperbola from its equation.*

Resuming (II.) we have $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$.

$$\text{Let } x = 0 \quad \therefore\ y = \pm b\sqrt{-1},$$

therefore the hyperbola does not intersect the axis of y.

$$\text{Let } y = 0 \quad \therefore\ x = \pm a = C\,A \text{ or } C\,A'.$$

Let $x < \pm a$, then the values of y are imaginary, and no part of the hyperbola lies between A and A'.

As x increases positively from a to ∞, the two values of y increase from zero to ∞, and give the branch $Z\,A\,Z'$.

Since, when the sign of x is changed the values of y do not alter, the negative values of x will give a branch of the hyperbola $z\,A'\,z'$ similar to the former on the other side of $B\,B'$.

The hyperbola therefore consists of two branches extending indefinitely to the right of A and the left of A', and symmetrically placed with respect to the axis $X'\,A\,X$.

304. Since $\frac{m}{e-1}$ or $\frac{A\,S}{e-1} = a \quad \therefore\ m$ or $A\,S = a\,(e-1)$,

$$\text{also } m : A\,L :: e : 1 \quad \therefore\ A\,L = \frac{m}{e} = \frac{a\,(e-1)}{e}.$$

The distance between the centre and the focus is,

$$C\,S = C\,A + A\,S = a + a\,(e-1) = a\,e, \text{ and } \frac{C\,S}{C\,A} = e,$$

which is called the eccentricity.

305. If in equation (III.) $a^2\,y^2 - b^2\,x^2 = -a^2\,b^2$, we change x into y, and y into x, which is equivalent to reckoning the abscissas along $C\,B$, and the ordinates along $C\,X$, we have, interchanging a and b, $b^2\,x^2 - a^2\,y^2 = -a^2\,b^2$,

$$\text{whence } y^2 = \frac{b^2}{a^2}(x^2 + a^2) \quad \ldots\ldots\ldots\ldots \quad (\text{V.})$$

the equation to an hyperbola of which $B\,B'$ is the transverse and $A\,A'$ the conjugate axis, and which relatively to the former is called the *conjugate* hyperbola.

306. From the vertex A, fig. XI., draw $A\,E$, $A\,E'$ perpendicular to $A\,C$, and each equal to $C\,B$, the conjugate axis; join $C\,E$, $C\,E'$, and produce the lines $C\,E$, $C\,E'$ indefinitely both ways; these lines continually approach but never meet the curve, and are called *asymptotes.*

307. *To express the distance of any point in the hyperbola from the focus in terms of the abscissa.*

By definition $S\,P = e\,.\,P\,M = e\,(C\,N - C\,L)$,

$$\text{but } C\,L = C\,A - A\,L = a - \frac{a\,(e-1)}{e} = \frac{a}{e},$$

$$\therefore\ S\,P = e\left(x - \frac{a}{e}\right) = e\,x - a.$$

Similarly it may be shown that $H P = e x + a$.

Subtracting, we have $H P - S P = 2 a$ = the *major axis*.

308. The equation to the tangent, found precisely in the same manner as that for the ellipse, is

$$y - y' = \frac{b^2}{a^2} \cdot \frac{x}{y'}(x - x');$$

and *therefore* (see 271) the equation to the normal will be

$$y - y' = -\frac{a^2}{b^2} \cdot \frac{y'}{x'}(x - x').$$

309. The *latus rectum*, (D D, fig. IX.), or double ordinate passing through S, is thus found,

$$x = m \quad \therefore y^2 = \frac{b^2}{a^2}(2 a m + m^2), \quad \text{and } y = \frac{b}{a}\sqrt{2 a m + m^2}.$$

⁂ The most important of the conic sections is the ellipse, as it very closely resembles a planet's orbit. If the planets were only attracted by the sun, their orbits would be perfect ellipses. The theorems deducible from this curve are described as countless.

Algebraical proofs of most of the following can be found in either Salmon's or Todhunter's *Treatise on the Conic Sections*. They have been selected for the purpose of comparison, and are such as are identical or similar for the ellipse and the hyperbola. The letters refer to fig. IX.

1. The tangent P T bisects the angle made by S P and the continuation of H P. It is termed the *external* bisector of the angle between the focial radii.

2. The normal P R bisects the angle between the focial radii P S and H P.

3. If the figure of the ellipse be altered by changing the foci S, H, without moving A, A', all the tangents drawn from the line N Q will meet in T.

4. The perpendicular dropped from the focus upon the tangent P T, meets the tangent in a point of the circle A Q A'.

5. The parallelogram, $P E F R$, is always equal to the rectangle $B C$, $C A$.

If R F be parallel to the tangent at P, then P R will be also parallel to the tangent at F.

P R is the *semidiameter* of the point P, and R F drawn parallel to the tangent, is the *conjugate semidiameter*, or *semiconjugate* of P R.

The circle, A Q A', is a form of the ellipse, which occurs when the major and minor axes are equal; the points S and H then meet in C, and every normal passes through C.

In the *ellipse* the square on the *ordinate* $P\ N$, is *less* than the rectangle under the *parameter* $A'\ N$ and *abscissa* $A\ N$.

In the *hyperbola*, the square of the ordinate *exceeds* this rectangle, and in the *parabola* these quantities are *equal*. From this deficiency, excess, and equality or similarity, the names ellipse, ἔλλειφις, hyperbola, ὑπερβολὴ, and parabola, παραβολή, were first given.

⁎ The hyperbola is of the least practical value of any of the conic sections. For the comparison of its properties with those of the ellipse, the following theorems have been chosen. The letters have reference to fig. XI.:—

1. The tangent $P\ T$ bisects the angle made by $P\ S$ and $H\ P$.
2. The normal being perpendicular to the tangent, is the *external* bisector of the angle between the focial radii, or it bisects the angle made by $P\ S$ and the production of $H\ P$.
3. If any number of hyperbolas be drawn having the same centre C, and the same major axis $C\ A$, and ordinates $N\ P$, $N\ P'$, &c., be drawn to the same abscissa $C\ N$, then the tangents at P, P', &c., will all meet the axis in the point T.
4. Any two such ordinates will always be in proportion to the minor axes to which they belong.
5. The perpendicular dropped from the focus upon a tangent $P\ T$ meets the tangent in a point of the circle whose centre is C and radius $C\ A$.
6. The parallelogram $P\ C$, $F\ R$ is always equal to the rectangle $C\ A$, $E\ B$.

$C\ A$ is the major semi-axis of the hyperbola passing through A and A', and $C\ B$ its minor semi-axis. Conversely $C\ B$ is the major semi-axis of the conjugate hyperbola passing through B and B', and $C\ A$ its minor semi-axis.

$C\ F$ being drawn parallel to the tangent $R\ T$, F is called the conjugate to P, and $C\ F$ is the conjugate semi-diameter to $C\ P$.

The hyperbola is said to be equilateral when the major and minor axes are equal; the asymptotes are then at right angles to one another, and the hyperbola and its conjugate are similar and equal.

EXAMINATION PAPERS:

FROM

1838.

MATRICULATION PASS EXAMINATION.

Monday, Nov. 5th.—Examiners,—Mr. Jerrard and Mr. Murphy.

1. Give the definition of a fraction, and explain the method of reducing fractions to a common denominator. Of the fractions $\frac{2}{3}, \frac{5}{7}$, which is the greater?

2. Add together the fractions $\frac{11}{17}, \frac{31}{51}, \frac{266}{357}, \frac{5}{13}, \frac{24}{39}$; and divide $43\frac{3}{7}$ by $7\frac{5}{8}$.

3. Reduce the fraction $\frac{4015}{6305}$ to its lowest terms, and transform it to a decimal.

4. Prove the rule for the multiplication of decimals. Multiply ·576 by 83·4, and divide 222·027 by ·0013.

5. Extract the square root of 9,512,295,961.

6. Find the simple interest on £207 12*s.* 6*d.* for $21\frac{1}{2}$ years, at 3 per cent. per ann,

7. Prove that $a\,b = b\,a$. State and demonstrate the truth of the rule for the signs in algebraic multiplication and division.

8. Multiply $x^5 + a^5 - a\,x\,(x^3 + a^3)$ by $(x^3 + a^3) + a\,x\,(x + a)$.

9. Find the greatest common measure of $x^5 - y^5$ and $x^2 - y^2$; and reduce to its lowest terms the fraction $\frac{3\,x^2 + 12\,x + 9}{x^5 + 5\,x^3 + 6}$.

10. Solve the following equations:

$$(1)\quad \frac{1 + x}{x + 5} = \frac{3 + x}{7 + x}; \qquad (2)\quad 5\,x + 3 = 11\,x - 21;$$

$$(3)\qquad 19\,x + \frac{1}{2}\,(7\,x - 2) = 4\,x + \frac{35}{2};$$

$$(4)\qquad \frac{4}{x + 2} + \frac{7}{x + 3} = \frac{37}{x^2 + 5\,x + 6}.$$

11. Give some account of the different methods employed for eliminating one of the unknown quantities in equations of the form

$$a\,x + b\,y = c$$
$$a'x + b'y = c'.$$

12. Find the values of x and y in the following systems of equations:

$$(1)\quad \begin{cases} x^3 + y^3 = 65 \\ x + y = 5 \end{cases} \qquad (2)\quad \begin{cases} \dfrac{3}{x} + \dfrac{3}{y} = 6 \\ x + y = 2 \end{cases}$$

$$(3)\quad \begin{cases} x + \sqrt{(x^2 - y^2)} = 8 \\ x - y = 1 \end{cases}$$

13. Explain when four quantities are said to be in proportion. Show that

$$\text{If } a : b :: c : d$$
$$\text{then } a + b : a - b :: c + d : c - d$$
$$\text{and } a^n : b^n :: c^n : d^n.$$

14. The sum of n terms of the arithmetic progression 1, 3, 5, 7, &c., is n^2. Prove this, and find the sums to infinity of the two geometric series, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$, &c.

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16}, \text{ \&c.}$$

15. At what time next before 12 o'clock are the hour and minute hands of a watch together?

1838. *Wednesday, Nov. 7th.*—Examiners,—Mr. JERRARD and Mr. MURPHY.

9. What is the area of a rectangular court, of which the length is 250 yds. 1 ft. 6 in., and the width 32 yds. 2 in.?

10. Find the value of the circulating decimal ·01750175 . . . .

11. Prove that $a^m . a^n = a^{m+n}$.

12. If S vary as A when a quantity B is given, and vary as B when A is given, generally it must vary as their product $A\ B$.

13. Prove the rule for finding the greatest common measure for arithmetic and algebraic quantities.

14. How many terms of the natural numbers, commencing with 4, give a sum 5350?

15. Find a multiplier which will rationalize $\sqrt{a} + \sqrt{b} + \sqrt{c}$, and give the value of $\dfrac{2 + \sqrt{5} + \sqrt{20}}{1 + \sqrt{45}}$ as far as 3 decimal places.

16. Of the external angles of a polygon, one-half are equal to those of a regular polygon of m sides, the others to those of a regular polygon of n sides. Find the number of sides of such a polygon.

1839. *Monday, Oct. 7th.*—Examiner,—Mr. JERRARD.

1. Find the difference between $\frac{1}{2} + \frac{1}{4} + \frac{1}{6}$, and $\frac{1}{3} + \frac{1}{5} + \frac{1}{7}$.

2. What are decimal fractions? How does the use of them facilitate calculation? Determine the form of fractions which may be converted into finite decimals.

3. Extract the square root of 16129, and of ·00015129; and give an explanation of the rule.

4. Find the simple interest on £587 10*s.* 6*d.* for 7½ years, at 3½ per cent. per annum.

5. Explain the rule for the multiplication of algebraical quantities.

6. Find the expansions of $(a + b)^2 (a - b)^5$, and divide $x^3 - y^2$ by $x^{\frac{1}{2}} - y^{\frac{1}{3}}$.

7. Reduce to their most simple forms,

$$(1)\ \frac{329}{423}, \qquad (2)\ \frac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2}.$$

8. Solve the following equations:

$$(1)\ x = 3x - \frac{1}{2}(4 - x) + \frac{1}{3},$$

$$(2)\ (x + 1)^2 = \{6 - (1 - x)\}\,x - 2,$$

$$(3)\ \left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 1, \\ \frac{1}{x} + \frac{1}{z} &= 2, \\ \frac{1}{y} + \frac{1}{z} &= \frac{3}{2} \end{aligned}\right\}$$

9. Insert six arithmetic means between 1 and 29.

10. If $\frac{a}{b} = \frac{c}{d}$, prove that $\frac{a + b}{a - b} = \frac{c + d}{c - d}$, or generally $\frac{a + m b}{a + n b} = \frac{c + m d}{c + n d}$, for all values of m and n.

11. Investigate the general expression for the sum of n terms of the geometric series $a + a r + a r^2 + \ldots$, and find the sum of the series $\frac{1}{5} - \frac{1}{25} + \frac{1}{125} -$, &c., *ad infinitum.*

12. What are the criteria of the divisibility of a number by 5, 9, and 11, respectively?

13. Find the least multiplier which will render 3234 a perfect square.

1839. *Wednesday, Oct. 9th.*—Examiner,—Mr. MURPHY.

11. Solve the following simple equations:

$$(1)\ \frac{2x + 5}{13} + \frac{40 - x}{8} = \frac{10x - 427}{19}.$$

$$(2)\ a x + b = a' x + b'.$$

(3) $\dfrac{25\,x^2 - 16}{10\,x + 8} = 3\,.\,\dfrac{x^2 - 4}{2\,x - 4}$ (by actual division).

(4) $\dfrac{a\,x^2 + b\,x + c}{a'\,x^2 + b'\,x + c'} = \dfrac{a\,x + b}{a'x + b'}$.

12. Find the values of the unknown quantities x, y, &c., in the following systems of equations:

$$(1)\quad \left.\begin{array}{c} \dfrac{11\,x - 5\,y}{22} = \dfrac{3\,x + y}{32} \\ 8\,x - 5\,y = 1. \end{array}\right\}$$

$$(2)\quad \left.\begin{array}{r} 7\,x - 3\,y = 1 \\ 4\,z - 7\,y = 1 \\ 11\,z - 7\,u = 1 \\ 19\,x - 3\,u = 1 \end{array}\right\} \qquad (3)\quad \left.\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 \\ \dfrac{x}{a'} + \dfrac{z}{c} = 1 \\ \dfrac{y}{b'} + \dfrac{z}{c'} = 1 \end{array}\right\}$$

13. Show how the arithmetical rule for partnership may be deduced from simple equations.

1840. *Monday, Oct. 5th.*—Examiner,—Mr. JERRARD.

1. Explain fully the rule for the multiplication of fractions.

2. Extract the square root of 103041, and of ·0758329; and show on what principle the rule depends.

3. Find the value of 275 cwt., 3 qrs., 25 lbs., at £3 7*s.* 6¾*d.* per cwt.

4. In what time will £225 amount to £317 at 5 per cent. simple interest?

5. Prove that $a^m a^n = a^{m+n}$ for all values of m and n.

6. What is the value of $\dfrac{2\sqrt{18} + \sqrt{32}}{\sqrt{56} + 3\sqrt{14}}$ to three decimal places?

7. Explain the method of finding the greatest common measure of two algebraic quantities, and reduce

$$\frac{x^4 - 3\,y\,x^3 - 8\,y^2\,x^2 + 18\,y^3\,x - 8\,y^4}{x^3 - y\,x^2 - 8\,y^2 x + 6\,y^3}$$

to its simplest form.

8. Solve the equations,

$$(1)\quad \frac{x}{2} + \frac{x}{3} = 15. \qquad (2)\quad \frac{x + 1}{3} - \frac{x + 2}{4} = 9 + \frac{x + 3}{5}.$$

$$(3)\quad \sqrt{15 + x} = 3 + \sqrt{x}.$$

9. Find the values of the unknown quantities in the equations,

$$(1)\quad \left.\begin{array}{r} 3\,x - 2\,y = 7 \\ 7\,y - 5\,x = 115 \end{array}\right\} \qquad (2)\quad \left.\begin{array}{r} x + y = 12 \\ x + z = 20 \\ y + z = 22 \end{array}\right\}$$

10. If two persons together can perform a piece of work in $7\frac{1}{2}$ hours, and one of them alone in 10 hours; how long will the other take to do it alone?

11. Define proportion, and prove that if four quantities are proportional the product of the extremes is equal to that of the means.

12. How many terms of the series 1, 3, 5, . . . will amount to 5041?

13. Find the least whole number which, when divided by 17, shall leave a remainder 7, but when divided by 26 the remainder 13.

1840. *Thursday, Oct. 8th.*—Examiner,—Mr. MURPHY.

9. Find the greatest common measure of $x^3 - 75x + 250$, and $x^4 - 50x^2 + 625$.

10. If two machines, working separately, perform a work respectively in a and in b hours, in what time will they perform it jointly, working with equal power as before?

11. Simple equations to be solved,

$$(1)\quad 4x + 3 = 13x - 150.$$

$$(2)\quad \frac{120}{x} + 18 = \frac{360}{x} - 6.$$

$$(3)\quad \frac{5}{x} + \frac{6}{x+1} = \frac{11}{x+2}.$$

12. Explain the nature of the following equations, and give the correct answers to them:

$$(1)\quad \frac{3x}{5} + 6 = \frac{15x-1}{25},$$

$$(2)\quad \frac{x-2}{4} = \frac{3x-2}{12} - \frac{1}{3}.$$

$$(3)\quad \frac{4}{(2x-1)(2x-5)} = \frac{1}{2x-5} - \frac{1}{2x-1}.$$

13. Solve the following equations between two or more unknown quantities:

$$(1)\quad 3x - y = 3,\quad 9x - 5y = -45.$$

$$(2)\quad xy = 42,\quad xz = 48,\quad yz = 56.$$

$$(3)\quad \left.\begin{aligned} 12x - 15y + 2z &= 13 \\ 7x + 4y - 8z &= 0 \\ \frac{x}{2} + \frac{y}{3} - z &= -2 \end{aligned}\right\}.$$

1841. *Monday, Oct. 4th.*—Examiner,—Mr. JERRARD.

1. What is a decimal fraction? Show how to multiply, and to divide, 6·012 by ·00345.

2. Find the simple interest of £3757. for $5\frac{1}{2}$ years, at $3\frac{1}{2}$ per cent. per annum.

3. If seven oxen eat an acre of grass in six days, how long will it take seventeen oxen to eat thirty-four acres?

4. Extract the square root of 4498641, and that of ·04498641.

5. Multiply $x + y$ by $x - y$, and give an interpretation of the result. Also expand

$$(x + y)^3, \text{ and } (1 + x)(1 - x - 3x^2 + 4x^3).$$

6. Compare the process of division in arithmetic with that in algebra. Divide

$$x^4 - 10x^3 + 35x^2 - 50x + 24 \text{ by } x^2 - 5x + 4,$$

$$\text{and } 21x^3 + x^2y - 28x^2z - 53xyz - 10xy^2 + 22y^2z + 44yz^2$$

$$\text{by } 7x^2 + 5xy - 11yz.$$

7. Show that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. Also show that if a be less than b, the fraction $\frac{a}{b}$ will be less than $\frac{a+x}{b+x}$ when x is any positive quantity.

8. Find the sum of n terms of the series $a + ar + ar^2 + ar^3 + \ldots$ and insert eleven arithmetic means between 7 and 151.

9. Solve the equations,

(1) $3 - (2x - 52) = 44 - x.$

(2) $\frac{x}{2} - \frac{x}{4} = 23 + \frac{x}{5} - \frac{x}{3}.$

(3) $\frac{x+1}{2} - \frac{3x-1}{7} = \frac{7x-11}{4}.$

(4) $10x + 14y = 19. \quad 102x - 66y = 29.$

What is the meaning of such equations as

(5) $59 + x = 25.$ (6) $\frac{2}{x^2-1} = \frac{1}{x-1} - \frac{1}{x+1}$?

10. A hare fifty leaps in advance of a greyhound, and taking three leaps for every two of the latter, is overtaken in 300 leaps of the greyhound: what is the proportion between the length of the leaps?

11. Every whole number may be reduced to the form,

$$a_n r^n + a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \ldots\ldots + a_1 r^1 + a_0,$$

in which r may be any integer whatever, and $a_n, a_{n-1}, \ldots\ldots a_0$ are integers, including zero, less than r.

1841. *Wednesday Morning, Oct. 6th.*—Examiner,—Rev. R. MURPHY.

8. Solve the following simple equations:

(1) $\frac{3x}{16} - 99\tfrac{3}{4} = 0.$

(2) $\frac{x}{5} + \frac{x}{6} - \frac{x}{3} = 1 + \frac{x}{2} - \frac{x}{8} - \frac{x}{10} - \frac{x}{4}.$

(3) $\frac{187 + x}{x - 37} = \frac{3x - 9}{3x - 121}.$

(4) $\frac{x}{a} + \frac{x + b}{a + b} + \frac{x + c}{a + c} = 3.$

(5) $\begin{cases} x + y = s, \\ x - y = d. \end{cases}$ (6) $\begin{cases} 3x + 5y = 370, \\ 5x + 3y = 590. \end{cases}$

9. *Problem.*—Ten years since, the ages of *A* and *B* together exceeded four times *C's* age by six years, but in ten years hence that sum will only exceed three times *C's* age by three years; when *C* was born *A* was exactly three times the age of *B*,—what are their present ages?

1842. *Monday, July 4th.*—Examiner,—Mr. JERRARD.

1. Give an account of our method of numeration. Wherein consists the great advantage it possesses over the methods of the ancients?

2. Upon what principles does the rule for the division of one whole number by another depend? What is meant by dividing one fraction by another?

3. Reduce

(1) $\frac{2}{3} + \frac{4}{5} + \frac{6}{7} - \frac{8}{9},$ (2) $\dfrac{1}{2 + \dfrac{3}{4 + \frac{5}{8}}},$

to their most simple forms; and divide ·0079968 by 2·24.

4. Investigate the rule for the extraction of the square root of a number. Take as an example the number 3392964. What is the square root of $3\tfrac{3}{4}$ to five places of decimals?

5. Define discount; and find the present value of £793 due six months hence, interest being at $3\tfrac{1}{2}$ per cent.

6. State the rule for the signs in the multiplication of algebraic quantities. Expand

(1) $(x + a)(x + b)(x + c),$ (2) $(x + a)^3 (x - a)^3;$

and show that

$$x^5 = \frac{(x + 1)^5 + (x - 1)^5}{2} - (10x^3 + 5x).$$

7. Divide $(x^3 - 6x^2 + 11x - 6)(x - 4)$ by $x - 1$. Also investigate the expression $\frac{a^m}{a^n} = a^{m-n}$. What is the meaning of the result, when m is equal to n?

8. Show that if

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots \text{ then will } \frac{a}{b} = \frac{a + c + e + \ldots}{b + d + f + \ldots}.$$

9. Solve the equations,

(1) $5x - 7 = 9(x - 1) - 42$.

(2) $\frac{x}{3} - \frac{x - 5}{7} = 11$.

(3) $(2a - x)(a + b) = 5a(b + x) - 7bx$.

(4) $\left.\begin{aligned} \frac{5x}{3} - 16 + \frac{2y}{5} &= 18 - y \\ \frac{x}{4} - \frac{7y}{2} - 1 &= 3(1 - x) \end{aligned}\right\}.$ (5) $\left.\begin{aligned} x + y + 2z &= 28 \\ 3x - 2y &= 33 \\ 2x + 5y - 7z &= 72 \end{aligned}\right\}.$

1842. *Wednesday, July 6th.*—Examiner,—Rev. R. Murphy.

9. Simple equations:—

(1) $\frac{5x + 7}{4} + 719 = 722x$.

(2) $\begin{cases} x + 2y + 3z = 14, \\ 3y - 7z = 15, \\ \frac{50z - 47x}{11y - 3} = \frac{108}{25}. \end{cases}$

(3) $17x - 20y = 30x + 18y - 7.$
$119x - 139y = 210x + 125y + 1.$

10. *Problem.*—What is the value of x, when £x and 18 shillings are twice the amount of £18 and x shillings precisely?

1843. *Monday, July 3rd.*—Examiner,—Rev. Prof. Heaviside.

1. Multiply 1756 by 304. Divide 43375 by 347. Explain the different steps of each operation.

2. Define a fraction. When is a vulgar fraction greater than unity? Find the value of the compound fraction $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$.

3. Reduce the fractions $\frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{7}{8}, \frac{9}{10}, \frac{11}{12}, \frac{13}{15}$, to equivalent fractions having a common denominator. Explain the rule you employ. Divide $\frac{121}{144}$ by $\frac{11}{12}$.

4. Add together $\frac{5}{8}$ of £1000 16*s.* 8*d.*, $\frac{3}{4}$ of £2400 12*s.* 4*d.*, $\frac{5}{7}$ of £3724 14*s.* 7*d.* What fraction of half a yard is 3 inches.

5. Represent as vulgar fractions 1·25, ·0004. How does it affect the value of a decimal to annex ciphers (1) after the decimal places, (2) between the decimal places and the decimal point. Decimals may be multiplied and divided by 10, 100, 1000, &c., merely by shifting the decimal point; show this. Divide ·000121 by 1·1.

6. Extract the square roots of (1) 119025, (2) 1·21, (3) 12·1.

7. Find the simple interest on £2475 12*s.* 4*d.*, at 3½ per cent. for five years.

8. Add $\frac{a-b}{2}$ to $\frac{a+b}{2}$. Multiply $x^2 - 2ax + a^2$ by $(x + a)$. Divide $x^2 - 7x + 12$ by $x - 3$, and $(x - y)$ by $x^4 - y^4$.

9. Define proportion. If four quantities be proportional, the product of the extremes is equal to the product of the means. What well known rule in arithmetic is founded on this property? If $a : b :: c : d$, prove $a + b : a :: \overline{c + d} : c$.

10. Solve the following equations:

(1) $3x - 7 = 4x - 10$. (2) $\frac{x}{2} + \frac{x}{3} = 13 - \frac{x}{4}$.

(3) $2x - 5 - \frac{x-3}{7} = x + 4$. (4) $\left.\begin{array}{l} 4x + 9y = 51 \\ 8x = 9 + 13y \end{array}\right\}$.

11. Two railroad trains start at the same time, one from London at the rate of 25 miles an hour, the other from Bristol at 30 miles an hour; the distance from London to Bristol is 120 miles; at what distance from Bristol will the trains meet?

1843. *Wednesday, July 5th.*—Examiner,—Mr. JERRARD.

9. Solve the equations,

(1) $\frac{x+2}{11} - \frac{x-7}{4} = 1 + \frac{x-10}{2}$, (2) $\left.\begin{array}{r} 5x + 6y = 137 \\ 13x - 4y = 23 \end{array}\right\}$;

and find the sum of the series,

$$a - \frac{a}{r} + \frac{a}{r^2} - \ldots \text{ to } n \text{ terms.}$$

1844. *Monday, July 1st.*—Examiner,—Rev. Prof. HEAVISIDE.

1. Explain concisely our method of numeration. Multiply 4765 by 205, and explain the successive steps of the process.

2. The numerator and denominator of a vulgar fraction may be

both multiplied or both divided by the same number without altering the value of the fraction: explain this. Reduce $\frac{7425}{8910}$ to its lowest terms.

3. Which is the greatest, $\frac{1}{19}$ of a pound sterling, or $\frac{1}{20}$ of a guinea? Express the difference between them. What fraction of £100 is £3 17*s.* 6*d.*?

4. If three-fourths of an estate be worth £535 10*s.*, what is the value of the whole of it?

5. What are decimal fractions? What are the advantages of representing fractions as decimals? Express $\frac{6}{625}$ as a decimal. What are *recurring* decimals? Show that if the fraction $\frac{6}{7}$ be reduced to a decimal, the digits *must* recur.

6. Explain the rules for *pointing* in the extraction of the square roots of whole numbers and decimals. Extract the square roots of 156·25, ·0064, ·064.

7. A person has £5635 stock, the annual interest on which is reduced from 3½ to 3¼ per cent.: what does he lose in income by the reduction, and what is his income after it?

8. Add $\frac{3a - 2b}{4}$ to $\frac{5a - 3b}{7}$.

Reduce $(7x - 2y) + (3x - 4y) - (9x - 5y)$.

Multiply $x^4 + 2x^3a + 4x^2a^2 + 8xa^3 + 16a^4$ by $x - 2a$.

Divide $16a^4 - 49b^4$ by $4a^2 + 7b^2$, and $x + y$ by $x^{\frac{1}{5}} - y^{\frac{1}{5}}$.

9. Solve the following equations:

(1) $3x - 5 = 2x + 7$. (2) $\frac{3x - 1}{2} - \frac{x + 1}{3} = 5$.

(3) $\frac{17 - 3x}{5} - \frac{4x + 2}{3} = 5 - 6x + \frac{7x + 14}{3}$.

(4) $5x + 4y = 58$,
$3x + 7y = 67$.

10. When are four quantities said to be proportional?

If $a : b :: c : d$, prove $a^2 : b^2 :: c^2 : d^2$. Find a fourth proportional to ·0004, 1·4, ·002.

11. At an election three candidates stood, and 1000 votes were polled. The second candidate on the poll had twenty votes more than the third; but if the first had only polled the same number of votes as the second, and the same number of votes had been polled, the third candidate would have been at the head of the poll by 10 majority. How many votes were polled by each?

1845. *Monday, July 7th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. Multiply £1364 17*s.* 6½*d.* by 19. Divide 855 by 57, and explain the different steps of the operation.

2. Explain the rule for reducing the fractions $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$, to equivalent ones having a common denominator. What is the *least* common denominator of the fractions equivalent respectively to

$$\frac{2}{3}, \frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{13}{15}, \frac{11}{20}, \frac{17}{21}?$$

3. What fraction of a pound sterling is 7*s.* 6½*d.*? Find the value of $\frac{5}{9}$ of £729 18*s.* 9*d.* Express in seconds $\frac{7}{13}$ of a year, taking the length of the year as 365 days, 6 hours.

4. By the last census, the population of Great Britain was found to be about 18,844,000, the parts employed in agriculture, and trade and manufactures, were respectively 1,499,000 and 3,110,000. How much per cent. of the whole population was each of these classes?

5. What are the advantages of decimal fractions? Express as a decimal, 17359 divided by one million. Divide ·00125 by 2·5. If the number of decimal places in the divisor exceed the number in the dividend, how do you proceed? Explain this, by making 2·5 the dividend and ·00125 the divisor.

6. A vulgar fraction may always be considered as expressing the division of the numerator by the denominator. Explain this, and hence show how to reduce such a fraction to a decimal.

7. In applying the rule for extracting the square root to a number which is not a perfect square, how do you explain the quantity obtained? Extract the square roots of 15876, ·0121, $\frac{144}{169}$, and the fourth root of 2·0736.

8. From $\frac{3a-2b}{4}$ take $\frac{2a-3b}{5}$.

Reduce (1) $x^2 + xy - (2x - xy) - (xy - x^2)$.

(2) $(x-3)(x+4)(x-5)$. (3) $\frac{x^6 - 64}{x-2}$.

(4) $(x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.

9. Solve the following equations:

(1) $\frac{x}{2} = 150 - 2x$.

(2) $\frac{3x+1}{2} - \frac{2x}{3} = 10 + \frac{x-1}{6}$.

(3) $15 - \frac{72}{x} = \frac{38}{x} - 7$.

(4) $\left.\begin{aligned} \frac{x}{9} + \frac{y}{8} &= 43 \\ \frac{x}{8} + \frac{y}{9} &= 42 \end{aligned}\right\}$.

10. Show how to find the sum of n terms of an arithmetical progression. Find the sums of the following series:

(1) The first fifty even numbers.

(2) 19, 14, 9, &c. to 16 terms.

(3) $\frac{4}{5}$, 1, $\frac{6}{5}$, &c., to 9 terms.

11. Define "ratio" and "proportion." If $a : b :: b : c$, prove $a : c :: a^2 : b^2$. If $ax - by = cy - dx$, find the ratio of x to y.

1846. *Monday, July 6th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. From £134 17*s.* 3*d.* take £98 16*s.* 10*d.*, and explain the successive steps of the operation.

2. Define a vulgar fraction. How does it appear that the numerator and denominator of a fraction may be both multiplied or both divided by the same number, without altering the value of the fraction? Reduce to its lowest terms $\frac{403}{899}$.

3. Take any number consisting of six figures, and show, from the nature of our system of numeration, that it is or is not divisible by 9, according as the sum of its digits is or is not divisible by 9.

4. Which would be most productive to the state, an income tax levied at 3 per cent., or at the rate of seven pence in the pound? Calculate the tax at the latter rate on an income of £1755 10*s.*

5. Explain the notation of decimal fractions. How does it affect the value of the decimal to shift the decimal point two figures, (1) to the right, (2) to the left? Express as vulgar fractions ·75, ·0005.

6. What vulgar fractions always produce decimals that terminate? Show that in reducing a fraction to a decimal, if the decimal does not terminate the figures will recur. Reduce $\frac{5}{7}$ to a decimal.

7. Extract the square root of 12·3201. Explain the rule for pointing the decimals.

8. How are the following terms used in algebra:—*Coefficient, exponent, factor, like quantities?*

From $\frac{a + b - 2c}{3}$ take $\frac{2a - b + c}{4}$. Reduce $\frac{a + b}{a - b} + \frac{a - b}{a + b}$.

Multiply $a^2 - 3ax + 4x^2$ by $a - 2x$.

Divide $8x^3 - 26x^2 + 11x + 10$ by $(2x - 5)$, and $x - y$ by $x^{\frac{1}{5}} - y^{\frac{1}{5}}$.

9. Solve the following equations:

(1) $30x - 500 = 340 - 40x$. (2) $\frac{x}{5} + \frac{x}{8} = 17 - \frac{x}{10}$.

(3) $\frac{7x + 5}{3} - \frac{16 + 4x}{5} + 6 = \frac{3x + 9}{2}$.

(4) $\left.\begin{array}{l} 12x + 13y = 37 \\ 17x - 19y = 15 \end{array}\right\}$.

10. Define the terms, *ratio*, *duplicate ratio*, *proportion*. When is one quantity said to vary inversely as another?

If $a : b :: \frac{1}{c} : \frac{1}{d}$, prove $\frac{a}{d} = \frac{b}{c}$. If $a : b :: c : d$, prove $a^3 : b^3 :: c^3 : d^3$.

11. A railroad 20 miles long is on a gentle slope; a train can travel along it and return in 40 minutes, and it is found that the train performs 3 miles down the slope in the same time that it takes to perform 2 miles up it. Required the time it takes to go and to return, and the rate of travelling each way.

1846. *Wednesday, July 8th.*—Examiner,—Mr. JERRARD.

9. Find the sums of the following series, having investigated the general formulæ on which they depend:

(1) The first ninety odd numbers.

(2) The first ninety even numbers.

(3) $1 + 2 + 4 + \ldots\ldots$ to ten terms.

(4) $1 + \frac{4}{5} + \frac{16}{25} + \ldots\ldots$ *ad infinitum.*

Show that the results in (1) and (2) are included under the forms n^2 and $n^2 + n$. What is meant by the sum of an indefinite series?

1847. *Monday, July 5th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. Multiply 15 yards, 2 feet, 11 inches, by 17. Divide 851115 by 345. Write down the true value of the remainder, after one figure has been obtained in the quotient and the first subtraction has been completed.

2. If 30 guns in a battle kill 2100 men, how many men would 210 guns kill at the same rate?

3. What is meant by the greatest common measure of two numbers? Find the greatest common measure of 247 and 570,

and show that the method you employ must give the greatest common measure.

4. Explain the rules for the multiplication and division of vulgar fractions. If two proper fractions be multiplied together, the numerical value of the product is less than either of them: show this.

5. Distinguish between *discount* and *interest*. Find the interest on £1456 10*s*., for 3 years, at 4½ per cent. simple interest, and the discount on the same sum for the same time.

6. Express as vulgar fractions the decimals, ·5, ·037. Explain why adding any number of ciphers to the right hand side of a decimal does not alter its value. How does it affect the value of the decimal to interpose a cipher between the decimal point and the first digit?

7. Explain the rule for pointing in the extraction of the square root of a whole number. If a decimal be a perfect square the number of decimal places must be even: show this. Extract the square roots of 190096, ·0064, and ·081.

8. Add together,

(1) $(1 + 2x - 4y)$ and $(2 - 3x + 6y)$.

(2) $(a + b - c)^2$ and $(a - b + c)^2$.

(3) From $5x^2 + 6xy + y^2$ take $(4x^2 - 3xy + 2y^2)$.

Multiply $(a^2 - 2ab + b^2)$ by $(a^2 + 2ab + b^2)$.

$(x^{-\frac{3}{4}} + x^{-\frac{1}{2}} y^{-\frac{1}{4}} + x^{-\frac{1}{4}} y^{-\frac{1}{2}} + y^{-\frac{3}{4}})$ by $(x^{-\frac{1}{4}} - y^{-\frac{1}{4}})$.

Divide $x^2 + 10x - 600$ by $(x + 30)$.

$$\frac{x^2}{x^2 - 9} \text{ by } \frac{x}{x + 3}.$$

9. Solve the following equations:

(1) $18x - 14 = 11x + 16$.

(2) $\frac{x}{6} + \frac{2x}{3} - \frac{7}{2} + \frac{3x}{4} = x + 3 + \frac{1}{2}$.

(3) $\left.\begin{array}{r} 4x + 7y = 62 \\ 3y - 2x = 8 \end{array}\right\}$.

If $x : x + 3 :: 3 : 4$, find x.

10. A railroad train travels at the rate of 24 miles an hour; two hours after it has started, an express engine travelling at the rate of 40 miles an hour is sent to overtake it. After what time and what number of miles will the express come up with the train?

11. When are a series of numbers (1) in arithmetical progression, (2) in geometrical progression?

Find the sum of 12 terms of the series, 19, 27, 35, &c., of 14 terms of the series $\frac{7}{2}, \frac{5}{2}, \frac{3}{2}$, &c.; continue 6 terms both ways the series 3, 6, 12.

1848. *Tuesday, July 4th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. Divide £516 12*s.* 6*d.* into 12 equal parts. Multiply 64208 by 804, and explain the arrangement of the rows of figures you add together to obtain the complete product.

2. What length of carpet $\frac{3}{4}$ yard wide, will cover a rectangular room 36 feet long, and 27 feet 9 inches wide, and what will be the cost of the carpet at 4*s.* 9*d.* a yard?

3. Show that any number will be divisible by 8, if its three last digits are divisible by 8, and only then. Find the greatest common measure of the two numbers, 78624 and 102232.

4. Explain the principle upon which vulgar fractions are reduced to their equivalents, having a common denominator. When may the common denominator be less than the product of all the denominators; and how is it then determined?

$$\text{Ex. } \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{5}{9}, \frac{11}{12}.$$

5. Find the interest on £1374 10*s.*, at $4\frac{1}{2}$ per cent. simple interest for 6 years. Find the present value of the same sum due 3 years hence, at the same rate.

6. Express as decimal fractions, $\frac{13}{1000}$, $\frac{19}{625}$. What are recurring decimals? Show that if $\frac{5}{7}$ be reduced to a decimal the digits recur.

7. Extract the square roots of 15129 and of 5188·3209, and explain the rule for pointing both the whole numbers and the decimals.

8. Add together $7ax + 6by - 3cz$ and $6ax - 3by + 2cz$.

From $2a - 3b - a + b$ take $2a - 3b - (a + b)$.

Multiply $x^3 - 3x^2 + 4x - 7$ by $x - 3$, and $x^{-\frac{1}{2}} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-\frac{1}{2}}$ by $x^{-\frac{1}{2}} - y^{-\frac{1}{2}}$.

Divide $x^4 - 5x^3 - 10x^2 + 17x - 5$ by $x^2 - 7x + 5$.

9. Solve the following equations:

(1) $2x - 14 = x + 12$. (2) $x + \frac{3x - 5}{2} = 12 - \frac{2x - 4}{3}$.

(3) $\left.\begin{array}{l} 5x + 6y = 76 \\ 4x - 3y = 14 \end{array}\right\}$. (4) $\left.\begin{array}{r} 2x + y + z = 13 \\ x + 2y + 3z = 16 \\ 3x - 2y + 4z = 14 \end{array}\right\}$.

10. Show that if four magnitudes be proportional, the product of the extremes is equal to the product of the means. If the four magnitudes be four straight lines, how is this result interpreted?

If $a : b :: b : c$ prove that $a : c :: a^2 : b^2$.

11. Show how to find the sum of an arithmetic series of n terms. Find the sum of 18 terms of the series 12, 18, 24, &c., and of 12 terms of the series $\frac{6}{5}, \frac{4}{5}, \frac{2}{5}$, &c. Find a geometrical mean between 8 and 128.

1849. *Tuesday, July 3rd.*—Examiner,—Rev. Prof. HEAVISIDE.

1. A labourer's wages are 9*s.* 8*d.* per week, how much does he receive for 11 weeks' work? Assuming that there are 58,240,000 acres of land in the United Kingdom, how many square miles of land does it contain? Note.—640 acres make 1 square mile.

2. Show that any whole number, 765468, may be expressed by the sum of its digits multiplied by powers of 10. Prove also by general reasoning, that a number is divisible by 9 when the sum if its digits is divisible by 9, and only then.

3. Add together the fractions $\frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{11}{12}, \frac{19}{36}$, and explain why they must be first reduced to a common denominator. What fraction must $\frac{2}{3}$ be divided by to give a quotient $\frac{11}{12}$? Can more than one such fraction be found?

4. What fraction of the earth's diameter (7900 miles) is a mountain $4\frac{1}{2}$ miles high? By what fraction of an inch would the height of such a mountain be properly represented, on a globe of 18 inches diameter?

5. If the whole revenue of the country (£50,000,000) were paid as interest on the national debt, (£760,000,000,) how much per cent. would it give? Find the interest on £7650 10*s.* for five years, at $3\frac{1}{2}$ per cent., simple interest.

6. Multiply 2·564 by ·047, and divide ·00169 by ·013. Verify your results by putting the decimals in the form of vulgar fractions.

7. Reduce $\frac{17}{625}$ to a decimal; and explain why, in reducing a fraction to a decimal which terminates, the number of decimal places depends on the form of the denominator of the fraction, and not on that of the numerator. Extract the square root of 258368·89.

8. Add together $7ax - 3xy$, $4ax - 2xy$, $ax + 7xy$.
From $(a+b)^2$ take $(a-b)^2$.
Multiply $(a^4 - 3a^3x + 4a^2x^2 - 7ax^3 + x^4)$ by $(a^2 - 2ax + 3x^2)$.
Divide $x^2 - 5x + 6$ by $(x-3)$, and $x^{-1} - a^{-1}$ by $x^{-\frac{1}{2}} - a^{-\frac{1}{2}}$.

9. Solve the following equations:

(1) $5x + 7 = 6x - 19 + 3x$. (2) $x + \frac{4}{5} - \frac{x-3}{2} = 6 + \frac{8}{10}$.

(3) $\left.\begin{array}{l} 3x - 4y = 22 - 2y \\ 5x + 2y = 42 \end{array}\right\}$. (4) $\left.\begin{array}{l} \frac{5}{x} + \frac{12}{y} = 2 \\ \frac{56}{y} - \frac{20}{x} = 5 \end{array}\right\}$.

10. When are four arithmetical quantities proportional? Find a fourth proportional to 7, 100, 756.

If $a : b :: c : d$, prove $\overline{a + b} : a :: \overline{c + d} : c$.

If $a : b :: c : d :: e : f$, prove $a + c + e : b + d + f :: a : b$.

11. When are magnitudes (1) in arithmetical, (2) in geometrical progression? Find an arithmetical and a geometrical mean between a and b; write down the 12th term of the series 7, 12, 17; and find the sum of five terms of the series $\frac{1}{3} - \frac{2}{9} + \frac{4}{27} - \&c.$

1850. *Monday, July 2nd.*—Examiner,—Rev. Prof. HEAVISIDE.

1. What is the cost of constructing a railroad 125 miles long, at the rate of £16 per yard?

2. What do you understand by the prime factors of a number? How may you determine by the inspection of the digits of a number when it is divisible by the numbers 2, 3, and 11 respectively? Find the greatest common measure of 7854 and 9768.

3. In Great Britain, the population of which is computed at 18,526,830, 5¼ per cent. of the population exercise the elective franchise: determine the number of electors. What per centage of the whole population of Great Britain is the population of Scotland, 2,620,180?

4. State, and explain, the rules for the multiplication and division of one vulgar fraction by another. Show that the multiplication of two proper fractions will give a product numerically less than either of them.

$$\text{Exs. } \frac{7}{8} \times \frac{11}{13}, \frac{2}{3} \div \frac{1}{2}.$$

5. Explain the notation of decimal fractions, and show how the value of a decimal is affected by moving the decimal point two places to the right or left. Write $\frac{375}{1000}$ as a decimal, and express the one-millionth part of the same fraction as a decimal. Multiply 85·345 by 4·173. Divide 25·6 by ·00016.

6. Reduce $\frac{17}{128}$ to a decimal. Write down the forms of the

denominators of vulgar fractions in their lowest terms, which will produce exactly six decimal places.

7. Explain the rule for pointing in the extraction of the square roots of whole numbers and decimals. Extract the square roots of 1194·3936, and of $\frac{14 \cdot 4}{16 \cdot 9}$.

8. When are quantities in algebra said to be *like* and *unlike?* What is the meaning of the index in a^6?

Add together $3\,x^2 - 2\,x\,y$, $4\,x^2 + 6\,x\,y$, $x\,y - x^2$.

From $\frac{a}{a - x}$ take $\frac{a}{a + x}$.

Multiply $27\,x^3 + 2\,x^2\,y + 3\,x\,y^2 + y^3$ by $3\,x - y$.

Divide $25\,x^2 - 30\,x\,y + 9\,y^2$ by $5\,x - 3\,y$.

9. Solve the following equations:

(1) $3\,x - 9 = 2\,x + 20.$ (2) $\frac{x - 3}{2} = \frac{x + 10}{3}.$

(3) $\frac{9 - 2\,x}{2} = \frac{3}{2} - \frac{7\,x - 18}{10}.$ (4) $\left.\begin{aligned} 7\,x + 8\,y &= 112 \\ 9\,x - 3\,y &= 51 \end{aligned}\right\}.$

10. Give an arithmetical definition of proportion. On what property of proportions is the "rule of three" founded? If 25 men do a piece of work in 15 days, in what time will 15 men do the same work?

If $a : b :: b : c$, prove (1) $\overline{a + b} : \overline{a - b} :: \overline{b + c} : \overline{b - c}.$
(2) $a : c :: a^2 : b^2.$

11. How are the terms, common difference and common ratio used in arithmetical and geometrical progression respectively? Form the arithmetical and geometrical series in which a is the first term, and 2 the common difference and common ratio. Write down the 20th term in the arithmetical, and the 6th in the geometrical series.

Find the sum of 15 terms of the series $9 + 24 + 39 +$ &c.

Find the limit of the sum of the series $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} +$, &c., *ad infinitum.*

1851. *Tuesday, July 1st.*—Examiner,—Rev. Prof. HEAVISIDE.

1. From 7503 take 871, and explain the process of "borrowing and carrying" in the common rule of subtraction.

2. Show that a number is divisible by 8 when its three last digits are divisible by 8, and only then. State and prove the criterion of divisibility by 9.

3. Distinguish between *interest* and *discount.* Find the simple interest on £3540 10*s.* for 3 years at 4½ per cent., and the discount on the same sum for the same time, at the same rate of interest.

4. Explain the principle upon which fractions are reduced to equivalent fractions, having a common denominator. Express by equivalent fractions with the least common denominator $\frac{5}{7}, \frac{11}{12}, \frac{13}{14}, \frac{14}{15}, \frac{17}{20}, \frac{19}{21}$. Reduce to its lowest terms, $\frac{5371}{59736}$.

5. Show how the common system of notation is extended to fractional quantities. Express 341·7205 in terms of its digits, multiplied or divided respectively by powers of 10. Hence obtain a rule for expressing a decimal by its equivalent vulgar fraction.

Multiply 72·14 by ·0376.

Divide 4230·1952 by 6·512, and 144 by ·00012.

6. Reduce $\frac{113}{3125}$ to a decimal. What are the conditions to be fulfilled, in order that a vulgar fraction may produce a finite decimal? If a vulgar fraction do not produce a finite decimal it must produce a recurring decimal.

7. Extract the square root of 423020·16. When are numbers said to be incommensurable? In extracting the square root of 13 by the ordinary rule, show that the process does not terminate.

8. How are the terms co-efficient, exponent, factor, used in algebra?

Add together $\left(\frac{a}{2} + \frac{b}{3} - \frac{c}{4}\right)$ and $\left(a - \frac{2\,b}{5} + \frac{c}{6}\right)$.

From $(7\,x^2 - 8\,x\,y + 9\,y^2)$ take $(5\,x^2 + 11\,x\,y + 8\,y^2)$.

Multiply $5\,x^3 + 7\,x^2\,y - 9\,x\,y^2 + 3\,y^3$ by $(2\,x^2 - 3\,x\,y + y^2)$.

Divide $3\,x^4 + 16\,x^3\,y - 33\,x^2\,y^2 + 14\,x\,y^3$ by $x^2 + 7\,x\,y$.

Reduce $\frac{x^3}{x^3 - 1} - \frac{x}{x - 1} + \frac{2\,x^2}{x^2 + x + 1}$.

9. Solve the following equations:

(1) $7\,x - 10 = 5\,x - 4.$

(2) $\frac{x}{3} - \frac{x}{4} + \frac{x}{6} = 3.$

(3) $\left.\begin{aligned} 3\,x + \frac{7\,y}{2} &= 22 \\ 11\,y - \frac{2\,x}{5} &= 20 \end{aligned}\right\}.$

(4) $\left.\begin{aligned} 2\,y - \frac{x + 3}{4} &= 7 + \frac{3\,x - 2\,y}{5} \\ 4\,x - \frac{8 - y}{3} &= 24 - x \end{aligned}\right\}.$

10. Define "ratio" and "proportion." Show from the algebraical mode of representing ratio, that magnitudes have the same ratio that their equimultiples have to each other.

If $a : b :: c : d$, and a the greatest magnitude, prove

(1) $\overline{a - b} : \overline{a + b} :: \overline{c - d} : \overline{c + d}.$

(2) $(a + d) > (b + c).$

11. When are three quantities, a, b, c, (1) in arithmetic, (2) in geometric progression? Express, for each progression, b in terms of a and c.

Find the sum of 19 terms of the series 7, 14, 21, &c.

Write down the sixth term of $\frac{2}{3}, \frac{2}{9}, \frac{2}{27}$, &c.

Find an arithmetic mean between $\frac{1}{2}$ and $\frac{1}{4}$, and three geometric means between $\frac{2}{3}$ and $\frac{2401}{24}$.

1852. *Tuesday, July 6th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. Divide £2561 4*s.* 6½*d.* by 7; explain briefly the steps of the operation.

2. When is one number said to be the greatest common measure of two others? Find the greatest common measure of 1575 and 3885, and explain why the process employed gives the greatest common measure.

3. Find the simple interest on £1210 10*s.* for 4 years, at 3 per cent. per annum.

The annual divisible receipts of a railway company are £437,500, and there are 250,000 shares at £21 each; what would be the dividend for each share, and what rate per cent. would be paid on each share?

4. Multiply $\frac{7}{10}$ by $\frac{39}{115}$. Explain the rule for multiplying two fractions, and show that the product of two proper fractions must always be numerically less than either of them.

If n be a whole number, what is the least value of n for which $(\frac{2}{3})^n$ is less than $\frac{1}{8}$?

5. What do the digits 5, 7, 3, represent, in the decimal ·573? Show that ciphers may be added without limit to *one* side of a decimal without altering its value: what change in the position of the decimal point increases a decimal one-hundredfold? Express as a decimal the millionth part of ·7.

Multiply 41·038 by 3·94.

Divide ·05265 by 13·5. Divide 25·6 by ·000016.

6. What are recurring decimals? Find the recurring decimal equivalent to $\frac{5}{7}$, and find the vulgar fraction equivalent to the recurring decimal ·81246246.

If the true value of a decimal be 3·14159, show that 3·142 is nearer the true value than 3·141.

7. Explain the rule for pointing in the extraction of the square root of whole numbers and decimals.

Extract the square roots of (1) 32·1489, (2) ·144.

8. When are quantities in algebra said to be "like" and "unlike?" Give examples.

Add together $(3a + 4b + 5c)$, $(2a - 9b + 6c)$, $(4a - 3b - 7c)$.

From $(a + b)^2$ take $(a - b)^2$.

Multiply $(x^3 - 7x^2 + 8x - 9)$ by $(x^2 - 2x + 1)$.

Divide $x^4 - 81y^4$ by $x - 3y$.

Explain the equation $x^8 \times x^{10} = x^{18}$.

9. Solve the following equations:

(1) $4x - 2 = 3x + 3$. (2) $\dfrac{x+1}{2} + \dfrac{x+2}{3} - \dfrac{5-x}{4} = 14$.

(3) $\dfrac{x-3}{x+2} = \dfrac{1}{2} + \dfrac{x-3}{2x-1}$. (4) $\left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 13 \\ \frac{x}{5} + \frac{y}{8} &= 5 \end{aligned}\right\}.$

10. (α) If $a : b :: c : d :: e : f$,

Prove (1) $a : b :: a + c + e : b + d + f$.

(2) $a : b :: c - e : d - f$.

(β) When are quantities in arithmetical progression? Write down 7 terms of the arithmetic series of which a is the middle term and b the common difference.

Find the sum of 20 terms of the series 3, 7, 11, 15, &c.

When are quantities in geometrical progression? a, x, y, b, are in geometrical progression, find x and y in terms of a and b.

1853. *Tuesday, July 5th.*—Examiner,—Rev. Prof. Heaviside.

1. In dividing one whole number by another, what does the quotient determine? Divide 243584 by 346, and explain the steps of the operation.

2. Show that any number will be divisible by 12, if its two last digits be divisible by 4, and the sum of its digits be divisible by 3 also. What are the prime factors of a number? resolve 54180 into its prime factors.

3. Find the simple interest on £4572 15*s*. for 9 years, at $4\frac{1}{2}$ per cent.

If the three per cent. stock be at 98, and the three and a quarter per cent. stock be at 101, which stock is it most advantageous to buy? What income will £5000 invested in the three per cent. stock produce?

4. Explain the principle upon which vulgar fractions are added together.

Add together $\frac{13}{15}$, $\frac{17}{20}$, $\frac{19}{21}$, $\frac{23}{25}$, $\frac{29}{30}$.

What fraction of a guinea added to 4*s.* 6*d.* is equal to 15*s.*? Is a proper fraction increased or diminished by adding the same number to its numerator and denominator?

5. Express as decimal fractions $\frac{7}{10}$, $\frac{13}{1000000}$. What is the distinction between decimals and whole numbers, as respects the prefixing and affixing ciphers to the right and left of the significant digits?

Divide ·365 by 20.

If in obtaining the quotient you cut off the cipher from the divisor and actually divide by 2, what corresponding change should be made in the dividend?

6. Perform the operations indicated below:—

(1) $36{\cdot}01 - 2{\cdot}987564$. (2) $2{\cdot}745 \times 45{\cdot}674$.

(3) $233{\cdot}8268 \div 3{\cdot}46$. (4) $6{\cdot}25 \div {\cdot}000125$.

(5) $\sqrt{2119{\cdot}6816}$.

Verify the result of (4) by vulgar fractions.

7. Why must the decimal equivalent to $\frac{5}{7}$ recur? Find that decimal. Find the vulgar fractions equivalent to the recurring decimals.

(1) ${\cdot}717171\dot{7}\dot{1}$. (2) ${\cdot}80654\dot{6}5\dot{4}$.

Find the value of ${\cdot}3333\dot{3}$ of $2\frac{1}{2}$ guineas.

8. State the rule of signs when one algebraical term is multiplied by another.

Add together $7\,x - 4\,y$, $3\,x + 5\,y$, $9\,x - y$.

From $(2\,a + 3\,b)^2$ take $(a - 2\,b)^2$.

Multiply $a^4 - 2\,a^3\,b + 2\,a^2\,b^2 - 3\,a\,b^3 + 2\,b^4$ by $a^2 - 2\,a\,b + b^2$.

Divide $(3\,x^2 - x - 10)$ by $(3\,x + 5)$.

Find $(x - 2\,a)^3$.

9. Solve the equations:

(1) $9\,x - 4 = 8\,x + 12$. (2) $\frac{x}{2} + \frac{x}{3} = \frac{x}{4} + 7$.

(3) $\left.\begin{aligned} 2\,x + 3\,y &= 32 \\ 11\,y - 9\,x &= 3 \end{aligned}\right\}$. (4) $\left.\begin{aligned} x + y &= 9 \\ x + z &= 10 \\ y + z &= 11 \end{aligned}\right\}$.

10. When are magnitudes in arithmetical progression, and when in geometrical?

Sum the series $4 + 11 + 18 + \ldots\ldots$ to 9 terms.

Sum the series $3 + 6 + 12 + \ldots\ldots$ to 6 terms.

What is the arithmetic mean between $2\,a - 3\,d$ and $2\,a + 5\,d$?

If $a : b :: b : c$, prove (1) $b^2 = a\,c$, (2) $a : c :: a^2 : b^2$.

1854. *Tuesday, July 4th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. Multiply £1365 17*s.* 11*d.* by 63.

Reduce 167805 ounces avoirdupois to tons, &c.

In reducing ounces to pounds, and pounds to quarters by steps of short division, explain the true values of the final remainders in each line of the process.

2. How may it be detected by the relations existing amongst the digits of any number, when that number is or is not divisible (1) by 3, (2) by 11. Find all the divisors of 2145.

3. What is the amount of £2674 15*s.* for four years at 5½ per cent. simple interest?

£7262 10*s.* invested, returns an income of £326 16*s.* 3*d.*, what rate per cent. is paid on the investment?

4. Define a vulgar fraction, and show from your definition that $\frac{2}{3} = \frac{4}{6} = \frac{10}{15}$.

Reduce the fraction $\frac{3503}{4294}$ to its lowest terms.

Can the greatest common measure of two numbers exceed the difference between them?

Express $\frac{7}{9}$ of half-a-guinea as the fraction of a crown.

What fraction of 12*s.* 6*d.* must be added to $\frac{5}{7}$ of a guinea to make a pound sterling?

5. Express as vulgar fractions ·5, ·0007, and as decimal fractions $\frac{625}{1000}$, $\frac{12}{1000000}$.

Perform the operations indicated below:—

(1) $4{\cdot}72 + 125{\cdot}5 - 54{\cdot}22$. (4) $1{\cdot}69 \div {\cdot}000013$.
(2) $22{\cdot}78 \times 3{\cdot}175$. (5) $\sqrt{1{\cdot}69}$.
(3) $72{\cdot}3265 \div 2{\cdot}278$. (6) $\sqrt{14{\cdot}4}$.

In Example (2) write down the *true value* of the product represented in each line of the operation.

6. Reduce $\frac{113}{625}$ to a decimal, and point out why the decimal terminates, and *what* determines the number of decimal places. Write down all the possible denominators of vulgar fractions in their lowest terms, which shall produce finite decimals of exactly *four* places.

Find the value of ${\cdot}876543\dot{6}54\dot{3}$.

Prove ${\cdot}333\dot{3} \times {\cdot}2121\dot{2}\dot{1} = {\cdot}0707\dot{0}\dot{7}$.

7. Explain the rule in algebra for removing a bracket which has a (—) sign before it.

Show that $(3a - 2b + 5c) - \{2a - 2b + 4c\} = a + c$.
Multiply $(2x^4 - 3x^3y + 4x^2y^2 - 6xy^3 - 2y^4)$ by $(3x - y)$.
Divide $(a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)$ by $(a^2 - 2ab + b^2)$.
Divide $(x^2 - y^2)$ by $(x^{\frac{1}{2}} - y)$.
Prove $(1 + x - x^2)^2 - (1 - x + x^2)^2 = 4x(1 - x)$.

8. Solve the following equations:

$$(1)\ 5x + 4 = 8x - 8.$$

$$(2)\ \frac{x}{5} + \frac{x}{2} - \frac{x}{12} = \frac{2x}{15} + 29.$$

$$(3)\ \frac{x+3}{6} - \frac{2x-7}{7} + 8 = \frac{x}{3}.$$

$$(4)\ \left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 29 \\ \frac{x}{3} - \frac{y}{2} &= 2 \end{aligned}\right\}.$$

9. Given the first term and the common difference of any arithmetic series, write down six terms of the series.

What is the 12th term of $7 + 12 + 17 +$ &c.?

What is the sum of 100 terms of $5 + 9 + 13 +$ &c.?

Insert six arithmetic means between 17 and 3.

Find the sum of five terms of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c.; to what limit does the sum approach as the number of terms is increased without limit?

10. Find a fourth proportional to 7, 63, ·015.

If $ax - by = cx + dy$, find the ratio $x : y$.

If $a : b :: c : d$, prove (1) $a^2 : b^2 :: c^2 : d^2$,
(2) $a^2 + b^2 : c^2 + d^2 :: a^2 : c^2$.

$$\text{If } \left.\begin{aligned} a : b &:: \alpha : \beta \\ b : c &:: \beta : \gamma \\ c : d &:: \gamma : \delta \end{aligned}\right\} \text{ prove } a : d :: \alpha : \delta.$$

1855. *Tuesday, July 3rd.*—Examiner,—Rev. Prof. HEAVISIDE.

1. How many half-crowns are there in £756 17*s.* 6*d.*? How many years are there in 7305 days, the length of the year being taken at 365¼ days? In the last example explain briefly the process employed.

2. What is a prime number? Find all the prime divisors of 25740, and show from the decimal system of notation, that any number is divisible by 9 when the sum of its digits is so divisible.

3. Find the simple interest for 3 years on £4564 15*s.*, at 4½ per cent.; and calculate the income-tax on the yearly interest at 1*s.* 2*d.* in the pound sterling.

At what rate per cent. is such an income-tax levied?

4. Define the least common multiple of two or more numbers. Show, with reference to the numbers 592, 1369, that the least

common multiple is equal to the product of the numbers divided by their greatest common measure.

Reduce $\frac{11}{15}, \frac{17}{20}, \frac{19}{21}, \frac{23}{28}$ to equivalent fractions having for a common denominator the least common multiple of their denominators.

By what fraction will $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ differ from unity?

What fraction of £24 is $\frac{3}{7}$ of £1 10*s.*?

5. Show how the common system of the decimal notation is extended to express fractional values.

Hence show how to express any decimal fraction as a vulgar fraction.

Express in the decimal notation (1) $\frac{3475}{100}$, (2) $\frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000}$.

Perform the operations indicated below:—

(1) $25{\cdot}72 - 18{\cdot}945$. (2) $45{\cdot}64 \times {\cdot}00035$.

(3) $443{\cdot}3 \div 12{\cdot}4$. (4) $16{\cdot}9 \div {\cdot}00013$.

(5) $\sqrt{5394{\cdot}9025}$.

In Ex. (1) explain the process of borrowing and carrying.

6. Reduce $\frac{13}{16}$ to a decimal.

Show why every vulgar fraction in its lowest terms, with 16 for its denominator, must be equivalent to a finite decimal of four places.

Find the fraction equivalent to the recurring decimal ${\cdot}47878\dot{7}\dot{8}$.

Prove ${\cdot}333\dot{3} - {\cdot}1212\dot{1}\dot{2} = {\cdot}212121\dot{2}\dot{1}$.

7. What is the coefficient of an algebraical quantity? Is any coefficient understood when none is expressed?

Reduce $(7x + 5y) + (2x - 3y) - (x - y)$.

Multiply $(7x^3 - 3x^2y + 2xy^2 - 6y^3) \times (3x - 4y)$.

Divide $(6a^3 - a^2b - 14ab^2 + 3b^3)$ by $(2a - 3b)$.

Reduce $1 - \frac{(a-b)^2}{(a+b)^2}$.

8. Solve the following equations:

(1) $5x - 11 = 3x + 13$. (2) $\frac{3x}{5} - \frac{4x}{10} + \frac{11x}{15} = \frac{24 - 2x}{3}$.

(3) $\left.\begin{aligned} 3x - 2y &= 11 \\ 4x + 7y &= 63 \end{aligned}\right\}$. (4) $\left.\begin{aligned} 2x - \frac{3x - y}{7} &= 15 + \frac{x}{2} \\ 3y - \frac{5x - 4}{8} &= \frac{x + y}{3} + 29 \end{aligned}\right\}$.

9. Find a third proportional to 5, ·05.

If $7x + 5y : 7x - 5y :: 31 : 11$, find $x : y$.

If $a : b :: c : d :: e : f$,

Prove (1) $\frac{a}{e} = \frac{b}{f}$. (2) $a : b :: (a + c + e) : (b + d + f)$.

10. Express six terms of the arithmetic series of which 5, 9, shall be the two middle terms.

Sum the series 11 + 16 + 21 + &c. to nine terms.

The sum of any number of terms of the series 1 + 3 + 5 + &c. is always a square number.

To what progression does the series following belong?

$$1 + \frac{1}{3} + \frac{1}{9} + \text{\&c.}$$

Find its sixth term, and its sum to five terms, and to infinity.

Insert two geometric means between ·9 and ·0009.

1856. *Tuesday, July 8th.*—Examiner,—Rev. Professor **Heaviside.**

1. Apply the "Rule of Three" to the solution of the following question:—

If 12 men can perform a piece of work in 8 days, in what time will 48 men perform the same?

Upon what property of numbers is this rule founded?

2. Find the simple interest on £2045 10*s.* for 7 years at $3\frac{1}{4}$ per cent.

At what rate per cent., simple interest, will £450 amount to £513 in four years?

3. When is one number the measure of another? When is one number the greatest common measure of two others? Give instances. Find the greatest common measure of 4212 and 13455. In the application of this rule, how is it indicated that the numbers are prime to each other?

4. State the rule for the division of one vulgar fraction by another. Divide $\frac{3}{2}$ by $\frac{4}{5}$; show that a proper fraction will always be increased by dividing it by another proper fraction. By what fraction must $\frac{11}{12}$ be divided to give a quotient 3?

The length of $\frac{1}{360}$ of the earth's circumference is $69\frac{1}{22}$ miles. What is the earth's diameter, assuming the diameter of a circle to be $\frac{7}{22}$ of its circumference?

5. Express $\frac{5}{10}$, $\frac{17}{100}$, $\frac{6}{10000}$, as decimal fractions.

Show from the notation of decimal fractions that

$$365{\cdot}74 = 3{\cdot}6574 \times 100.$$

Perform the operations indicated below:—

(1) 5·5 — 7·0354 + 2·14. (2) 13·754 × 2·453.

(3) ·31075 ÷ 1·13. (4) $\sqrt{51{\cdot}912025}$.

6. What are recurring decimals? Show that in reducing a vulgar fraction to a decimal, if the decimal does not terminate it must recur. Find the repeating period of $\frac{5}{7}$.

Find the vulgar fraction equivalent to the recurring decimal $\cdot 45656\dot{5}\dot{6}$.

7. What is the value of x^4 where $x = 5$?

Simplify $(3\,a + 5\,b - 2\,c) - (a + 5\,b - c)$.

Multiply $(a^4 - 3\,a^3\,b + 2\,a^2\,b^2 - 4\,a\,b^3 + b^4)$ by $(a^2 - 2\,a\,b + 3\,b^2)$.

Divide $(8\,x^3 - 2\,x^2 - 11\,x + 15)$ by $(2\,x + 3)$.

Prove $1 + \dfrac{b^2 + c^2 - a^2}{2\,b\,c} = \dfrac{(a + b + c)\,(b + c - a)}{2\,b\,c}$.

8. Solve the following equations:

(1) $63\,x - 119 = 58\,x + 1$. (2) $\dfrac{4\,x}{3} - \dfrac{5\,x}{7} = x - 8$.

(3) $\left.\begin{aligned} 3\,x - 7\,y &= 7 \\ 11\,x + 5\,y &= 87 \end{aligned}\right\}$. (4) $\left.\begin{aligned} \frac{x}{9} + \frac{y}{8} &= 43 \\ \frac{x}{8} + \frac{y}{9} &= 42 \end{aligned}\right\}$.

9. Express, by an equation, the relation, $a : b :: c : d$.

If $a : b :: c : d$,

prove (1) $m\,a : m\,b :: n\,c : n\,d$,

(2) $(a + b)^2 : (a - b)^2 :: (c + d)^2 : (c - d)^2$.

If $a : b :: b : c$, prove $a : c :: a^2 : b^2$.

Verify this last property when $a = 3$, $b = 6$.

10. Find the arithmetical and the geometrical means between 6 and 24.

Sum the series $9 + 14 + 19 +$ &c., to 12 terms.

$\frac{4}{5} + \frac{3}{5} + \frac{2}{5} +$ &c., to 7 terms.

$4 + 12 + 36 +$ &c., to 5 terms.

1857. *Tuesday, July 7th.*—Examiner,—Rev. Prof. HEAVISIDE.

1. If 22 yards = 1 chain, and 4840 square yards = 1 acre, how many square chains are there in 5 acres?

2. Find the simple interest on £1365 6*s*. 8*d*. for seven years at 3½ per cent.

At what rate of simple interest will £300 amount to £350 in seven years?

3. Write down any two numbers of which 205 is the greatest common measure. Find the greatest common measure of 3955 and 1808, and show why the process you employ necessarily gives the greatest common measure required.

Reduce the fractions $\frac{5}{6}$, $\frac{7}{12}$, $\frac{9}{15}$, $\frac{17}{20}$, $\frac{41}{60}$ to equivalent fractions with the *least* common denominator.

Could all these fractions be expressed with a denominator 360?

4. Show from general reasoning that the product of two fractions $\frac{5}{6}$ and $\frac{7}{8}$ must be arithmetically equivalent to the compound fraction $\frac{5}{6}$ of $\frac{7}{8}$.

What number is that of which the part expressed by $\{\frac{1}{3} + \frac{1}{4} + \frac{1}{6}\}$ is 45?

Reduce $3\frac{1}{7}$ of £4 11*s*. to the fraction of $13\frac{1}{3}$ guineas.

5. Show how the decimal system of notation is extended to express numerical magnitudes less than unity. What part of the unit is expressed by ·45 and by ·00007 respectively?

State the rule for the multiplication of decimals, and verify it by vulgar fractions in the case of the product $33 \cdot 175 \times \cdot 0008$.

Perform the operations indicated below:—

$$(1) \quad \cdot 00044408 \div \cdot 0112.$$

$$(2) \quad \sqrt{\frac{14 \cdot 4}{16 \cdot 9}}. \qquad (3) \quad \sqrt{3253 \cdot 5616.}$$

In Ex. (3), state the rule for pointing the decimals.

6. Reduce to a decimal $\frac{113}{625}$; why is the decimal finite? Reduce $\frac{625}{113}$ to a decimal; express the true value of the remainder in the division after four decimal places have been obtained. Find the value of the decimal $\cdot 84141\dot{4}\dot{1}$, and prove $\cdot 000999\dot{9} = \cdot 001$.

7. Explain the use of the bracket in algebra. If a negative sign stand before a bracket, how may the bracket be removed?

Simplify (1) $\frac{a^2 + 2\,a\,b - b^2}{2} - \frac{a^2 - 2\,a\,b + b^2}{4}$.

Multiply

$(a^5 - 5\,a^4\,b + 10\,a^3\,b^2 - 10\,a^2\,b^3 + 5\,a\,b^4 - b^5)$ by $(a^2 - 2\,a\,b + b^2)$.

Divide $(x^5\,y^5 - 1)$ by $(x\,y - 1)$.

Reduce $(1 + 2\,x + 3\,x^2)^2 - (1 - 2\,x + 3\,x^2)^2$.

8. Solve the equations:

$$(1) \quad x = 15\,x - 42. \qquad (3) \quad \frac{x}{2} - \frac{5\,x + 4}{3} - \frac{4\,x - 9}{3} = 0.$$

$$(2) \quad \frac{x}{2} + \frac{x}{3} = x - 7. \qquad (4) \quad \frac{x}{3} + \frac{y}{5} = 8.$$

$$\frac{x}{9} - \frac{y}{10} = 1.$$

If $x + 4 : 2\,x - 4 :: 2 : 3$, find x.

9. Insert two arithmetic means between 20 and 65. Write down 9 terms of the arithmetic series of which the three middle terms are 3, 11, 19. Find the fifth term of the geometric series 7, 21, 63.

Sum to 12 terms $8 + 13 + 18 +$ &c.

Sum to 13 terms $2 + 1\frac{7}{9} + 1\frac{5}{9} +$ &c.

Find the limit to the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c.

If $7a + b : 3a + 5b :: 7c + d : 3c + 5d$, prove $a : b :: c : d$.

1858. *Tuesday, July 6th.*—Examiner,—G. B. Jerrard, Esq.

1. Reduce $\frac{216}{288}$ to its lowest terms.

What is the product of 2·16 and 28·8? Prove the rule for the multiplication of decimals.

2. Extract the square root of 904401, explaining the process.

3. Find the simple interest on £547 15*s.* for 5 years at 3 per cent.

If a sum of money doubles itself in 40 years at simple interest, what is the rate of interest?

4. A ship sails with a supply of biscuit for 60 days, at a daily allowance of 1 lb. a head; after being at sea 20 days she encounters a storm, in which 5 men are washed overboard, and damage sustained that will cause a delay of 24 days, and it is found that each man's allowance must be reduced to five-sevenths of a pound. Find the original number of the crew.

5. If $X = \frac{a - b}{a + b}$, $Y = \frac{a + b}{a - b}$,

express

$$(1)\ X + Y, \qquad (2)\ X - Y,$$

in terms of a and b, in the most simple forms.

Give an account of the origin of negative quantities.

6. Solve the equations:

$$(\alpha)\quad 10 - 2x - 6 = 46 - 5x,$$

$$(\beta)\quad \frac{x - 5}{3} + \frac{x}{2} = x - \frac{x + 1}{8} - \frac{x - 1}{5},$$

$$(\gamma)\quad \left.\begin{aligned} 2x + 3y &= 61 \\ 5x - 4y &= 26 \end{aligned}\right\}.$$

7. What is the algebraical definition of ratio? How may we compare two or more ratios? Show that a ratio of greater inequality is diminished, and of less inequality increased, by adding the same quantity to each of its terms. Prove also that

if $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$.

8. Find the sum of a given number of quantities in arithmetical progression, the first term and the common difference being supposed known.

Ex. $\frac{1}{2}, -\frac{3}{4}, -2, \ldots\ldots$ to 24 terms.

What is the common difference when the first term is 1, the last 50, and the sum 204?

9. Insert n geometrical means between a and c; and show that their product will be $(a\,c)^{\frac{n}{2}}$.

1859. *Tuesday, July 5th.*—Examiner,—Rev. Professor HEAVISIDE.

1. If £24 7*s.* 10½*d.* be paid as income tax on an income of £650 10*s.*, what ought to be paid at the same rate on an income of £2450 6*s.* 8*d.*? and at what rate in the pound is the tax levied?

2. Find the simple interest on £3168 15*s.* for 4 years at 5½ per cent.

In how many years will £625 10*s.* amount to £813 3*s.* at 4 per cent. simple interest?

3. Explain the rule for the multiplication of one vulgar fraction by another, and show that it is consistent with the rule for the multiplication of whole numbers.

Multiply $\left(\frac{11}{12} - \frac{13}{15}\right)$ by $2\frac{2}{9}$.

The value of an ounce of standard gold is £3 17*s.* 10½*d.*: what fraction of one million sterling are 625 ounces of gold?

4. Show that any mixed number expressed as a decimal may be multiplied or divided by 1000 merely by shifting the decimal point.

Multiply 86·5 by ·00164. Divide 11·3 by 28·25.

Prove the first result by vulgar fractions.

5. Explain why the square of a decimal must have an even number of decimal places. Can a whole number with the digit 2 in the unit's place be an exact square?

Extract the square root of 1225·7001, and prove that the square root of $18{\cdot}\dot{7} = 4\frac{1}{3}$.

6. Find the values of the expressions $\left(\frac{x}{y} + \frac{y}{x}\right)$ and $\frac{x^3 - y^3}{x - y}$ when $x = 6$, $y = 4$; will the values be the same if $y = 6$, $x = 4$?

Perform the operations indicated in the following examples:—

(1) $\left(a^2 + 3\,b^2 - \frac{c^2}{2}\right) - \left(a^2 - 2\,b^2 - \frac{3\,c^2}{2}\right)$.

(2) $(a^3 + 3\,a^2\,x + 3\,a\,x^2 + x^3) \times (a^2 - 2\,a\,x + x^2)$.

(3) $(x^4 - 23\,x^2 + 18\,x + 40) \div (x^2 + x - 20)$.

(4) $(1 + x + x^2)^2 - (1 - x + x^2)^2$.

7. Solve the following equations:

(1) $\frac{x - 2}{3} - \frac{x - 3}{5} = \frac{x + 40}{12}$.

(2) $\frac{3x - 14}{4} - 5x - \frac{2x - 6}{11} = \frac{x}{2} - 72.$

(3) $(3x + 8) = 4y - 4 = 2(x + y - 1).$

8. Wishing to buy a certain number of railway shares, I found that if I bought the shares in the railway (A) which were at £40 a share, I should invest all my money; but if I bought the same number of shares in a railway (B) which were at £45 a share, I should not have money enough by £240. How much money had I to invest?

9. Find a fourth proportional to 1·5, ·09, ·45.

If $a : b :: c : d$ and $e : f :: g : h$, prove $a e : b f :: c g : d h$.

If $x^2 - y^2 : x^2 + y^2 :: 5 : 13$, prove $x : y :: 3 : 2$.

10. If a, $a + b$, $a + 2b$ are three consecutive terms of an arithmetic series, write down the three terms immediately preceding (a).

Sum the series $\frac{4}{5}$, $\frac{3}{5}$, $\frac{2}{5}$ to 7 terms.

Express the intermediate terms of the arithmetic series of which 4 is the first term, and 29 the sixth term.

In a geometrical series, if the common ratio be a proper fraction, what is meant by the sum of an unlimited number of terms having a limit? Give a numerical illustration.

Find a mean proportional to $\frac{a + b}{2}$ and $\frac{2ab}{a + b}$.

1860. *Tuesday, January* 10*th*.—Examiner,—Rev. Prof. HEAVISIDE.

1. By the payment of 2*s.* 1*d.* in London, a banker will give credit at Calcutta for one rupee; how many rupees may be received in Calcutta by the payment of £5025 16*s.* 8*d.* in London?

2. Show why it follows from our system of notation, that a number when divided by 9 leaves the same remainder as the sum of its digits will leave when divided by 9. Write down all the numbers that can be composed of the four digits 3, 4, 5, 6, which will each be exactly divisible by 11.

3. At what rate per cent. simple interest will £7433 6*s.* 8*d.* amount to £9942 1*s.* 8*d.* in 7½ years?

4. Show that the rule for dividing one fraction by another is consistent with the result of division in whole numbers.

Reduce to its equivalent single fraction the expression

$$\frac{\frac{2}{3} + \frac{4}{5} \text{ of } \frac{5}{9} - \frac{8}{21}}{1 + \frac{2}{3} \times \frac{3}{7} - \frac{5}{9}}.$$

5. Express as decimal fractions,

$$(1)\quad \frac{4}{10}+\frac{7}{1000}+\frac{8}{100000}.$$

$$(2)\quad \frac{11}{4}-\frac{17}{8}.\qquad (3)\quad \frac{13}{70}.$$

Could you tell, by inspection, that the last fraction would give a recurring decimal?

6. Reduce $\frac{(2{\cdot}05)^2 \times 2{\cdot}24}{{\cdot}0041}$; extract the square root of 529·092004.

7. Find the value of $x^2 - 5x + 7$ when $x = 3$; explain why $(x-4)^2$, and $(4-x)^2$, have the same value for any integral value of x. Simplify the following expressions:

$$(1)\quad 3a + 5b - \frac{c}{2} - \left\{a - \frac{2b}{3} + \frac{c}{4}\right\}.$$

$$(2)\quad (x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4) \times (x^2 - 2xy + y^2).$$

$$(3)\quad \frac{6x^3 - 19x^2y + 18xy^2 - 5y^3}{2x^2 - 3xy + y^2}.$$

8. Solve the following equations:

$$(1)\quad \frac{x-2}{5} - \frac{2x-7}{10} + 15 = \frac{16x-39}{10}.$$

$$(2)\quad \left.\begin{aligned} \frac{3x+2y-3}{4x-5y+16} &= \frac{9}{4} \\ 3x &= 5y \end{aligned}\right\}.$$

9. A courier undertakes to perform a journey on foot of 60 miles within 12 hours; he travels 6½ miles the first hour, but afterwards, in every successive hour, he travels ¼ of a mile less than in the preceding hour; will he perform his undertaking?

Find a third proportional to $\frac{2a^2b^2}{a^2+b^2}$ and ab.

If $a : b :: c : d$, prove $(a^2 + c^2)^{\frac{1}{2}} : (b^2 + d^2)^{\frac{1}{2}} :: a : b$.

1860. *Tuesday, July 3rd.*—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. A man bought £500 3 per cent. stock when consols were at 93½, and after having received one dividend, he sold at 96¾. What did he gain by the transaction?

What must be the rate per cent. that £73 may amount to £100 in 25 years?

2. Find the value of (1) ⅜ of 13*s.* 4*d.*, (2) ⅔ of 7*s.* 6*d.*, and (3) the recurring decimal ·11, &c., of 9*s.* 9*d.*; also, (4) express the first as a fraction of a guinea.

(5) Simplify $$\frac{\dfrac{1}{1-\dfrac{1}{2-\frac{1}{2}}} - \dfrac{1}{3} + \left(1 + \dfrac{1}{2}\right)}{3\left(1 + \dfrac{2}{3\frac{1}{2}}\right) - 4}.$$

3. Explain (1) the rule for the reduction of a recurring decimal to a fraction. (2) Reduce 2·4171717, &c., to a fraction.

(3) Divide ·04 by ·0003 and (4) 4 by ·003, and (5) raise ·25 to the power of ·5.

4. A grocer has equal quantities of two kinds of tea worth 5 shillings and 4 shillings per ℔. respectively. He takes one-third of each kind, and mixes it with the other. At what price per ℔. ought the two mixtures to be respectively sold?

5. Simplify the expressions

(1) $(2 - x + 3x^2) - \frac{1}{2}(4 - 2x + x^2)$.

(2) $$\frac{(1 - x^2)(1 - x^3)}{x(1 + x)(1 - x)^2} - \frac{x^3 + \dfrac{1}{x^3}}{x^2 + \dfrac{1}{x^2} - 1}.$$

(3) Find the factors of $1 - 2x + 3x^2$, and (4) determine the value of

$$\left(\frac{a - b}{b - c}\right) + \frac{1}{2}\left(\frac{5(a + c)}{2b}\right) - \frac{2b - c}{2}$$

when $a = 4$, $b = 2$, $c = 1$.

6. Solve the equations:

(1) $$1 + \frac{x}{2} - \frac{x}{3} = 4 - \frac{x + 1}{7} - \frac{x - 1}{5}.$$

(2) $$\left.\begin{aligned} \frac{x + 1}{y - 1} + \frac{y}{x - 4} &= 5 \\ \frac{x + 1}{y - 1} - \frac{3y}{x - 4} &= 1 \end{aligned}\right\}.$$

7. Extract the square roots (1) of 4·04010, and (2) of

$$(x^2 + 1)^2 + 4x(x^2 - 1).$$

Find what value of x will make $x^2 + 2ax + b^2$ the square of $x + c$. What does your result become when $a = b = c$?

If unity be taken from the square of any number, prove that the difference is equal to the product of two numbers, one greater by unity, and the other less by unity, than the original number.

8. When are four quantities said to be in proportion?

What value must be given to x to make $1 + x$, $2 + x$, $8 - x$, and $10 - x$, proportionals?

If $a : b :: c : d$, prove the equality

$$\frac{a^3 + b^3}{c^3 + d^3} \cdot \frac{b}{d} = \left(\frac{a + b}{c + d}\right)^4.$$

If 6 men can dig 14 yards per day of a trench 3 feet wide and 2 feet deep, how many men will be required to dig 12 yards in a day of a trench 7 feet wide and 6 feet deep?

9. Define an arithmetical and a geometrical progression.

If three numbers be in geometrical progression, must the middle one be increased or decreased that they may become an arithmetical progression?

Find the sum of the arithmetical progression

$$-\frac{1}{3}, -\frac{1}{12}, \frac{1}{6}, \text{ \&c., to 20 terms;}$$

and also of the infinite geometrical progression

$$1, \frac{1}{3}, \frac{1}{9}, \text{ \&c.}$$

1861. *Wednesday, Jan.* 16*th*.—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Find the least number which is divisible by all the numbers from 1 to 12.

A number may be multiplied by 625 by placing 4 ciphers at its right hand, and dividing by 16; explain the reason of this, and multiply 70354 by 625.

2. Simplify the expression

$$\frac{4\frac{1}{2} - 3\frac{1}{3} + 5\frac{1}{12}}{7\frac{1}{2} - 4\frac{1}{3} + 11\frac{1}{12}} - \frac{11\frac{3}{4} - 5\frac{7}{15}}{11\frac{3}{4} + 5\frac{7}{15}};$$

and prove that 17·975 of £71 2*s*. is equal to $\frac{1}{40}$th of £51120 18*s*.

3. Extract the square roots of 443556, ·0000004489, and $\cdot\dot{4}$.

Divide ·000456 by ·0000057, and calculate to 5 places of decimals the value of $\frac{1}{\sqrt{2}}$.

4. Define interest, discount, and present value, and find the present value of £911 13*s*. 3*d*. due 5 years hence at 3 per cent. simple interest.

If the $3\frac{1}{2}$ per cents. be at $95\frac{5}{8}$, and the 3 per cents. at 82, which is the better investment?

5. Add together $a + 3b + 5c$, $3a - 7b + 11c$, $4a - 5b - 15c$, and $a + 18b + 8c$; and multiply the result by the difference between $11a + 7c$, and $10a + 6c - b$.

Simplify the expression

$$\left(\frac{1 - x + x^2}{1 + 3x + 2x^2} - \frac{1 - 3x + 2x^2}{1 + x + x^2}\right) \cdot \frac{1 + 2x}{2 - x^2};$$

and resolve $x^4 + a^2x^2 + a^4$ into two quadratic factors.

6. Solve the equations

$$\frac{x-1}{4}-\frac{2x-3}{5}=11-\frac{5x-7}{20}.$$

$$x-\frac{c^2}{x-a}=a.$$

$$\left.\begin{array}{l} a(x+y)=xy \\ b(x-y)=xy \end{array}\right\}.$$

Determine for what value of x the expression $x^3 + 3cx^2 + a^2x + a^3$ will be equal to the cube of $x + c$; and prove that the difference between the cubes of the sum and difference of any two numbers is divisible by the sum of the square of the smaller number, and three times the square of the larger.

7. A person walking along the road in a fog, meets one waggon and overtakes another, which is travelling at the same rate as the former, and he observes that between the time of his first seeing and passing the waggons, he walks 20 yards and 60 yards respectively; find how far he can see in the fog; and compare his rate of walking with the rate at which each waggon is moving.

8. If $a : b :: c : d$, prove that

$$a + b : a - b :: c + d : c - d,$$

$$\text{and } a^2c + ac^2 : b^2d + bd^2 :: a^3 + c^3 : b^3 + d^3.$$

And also if $a + b + c + d : a + b - c - d :: b + d : b - d$, prove that

$$a : b :: c : d.$$

9. Having given the first term, the common difference, and the number of terms of an arithmetical progression, find an expression for its sum.

Find the sums of the series

$$29 + 28 + 27 + \ldots + 0 - 1 - 2 \ldots - 20,$$

$$\text{and } 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \ldots \text{ to infinity.}$$

Also find the sum of the series formed by taking every fourth term of the last series, beginning at the third.

1861. *Wednesday, July 3rd.*—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. How many yards of carpet, ¾ of a yard wide, will cover a room whose width is 20 feet, and length 22 feet?

By selling tea at 4*s*. 6*d*. per lb., a grocer cleared one-eleventh of his outlay, what would he have gained if he had sold the tea at 5*s*.?

2. Define a decimal. Explain and prove the rule for the division of one decimal by another. Express $\frac{2}{9}$ of 7*s*. 6*d*. + ·625 of 16*s*. as a decimal of £2 1*s*. 8*d*. Divide 2·88 by ·00016.

3. The periods of three planets which move uniformly in circular orbits round the sun, are respectively 200, 250, and 300 days. Supposing that their positions, relative to each other and to the sun, to be given at any moment, determine how many days must elapse before they have again exactly the same relative positions.

4. Simplify the quantities

$$\frac{4}{x-1} - \frac{3x+1}{x^2 - \frac{2}{3}x - \frac{1}{3}} \text{ and } \frac{x^3 + x^2 + x - 3}{x^3 - x - 6}.$$

Determine a quantity independent of x, which, being added to $a_0 + a_1 x + a_2 x^2 + \ldots$, will make it a multiple of $x - 1$.

5. Solve the equations

$$\frac{2x+1}{3} - \frac{3x-2}{4} = \frac{x-2}{6}. \qquad \left.\begin{aligned} xy + \frac{x}{y} &= 10 \\ xy^2 - x &= 6y \end{aligned}\right\}.$$

6. Define an arithmetical and a geometrical progression. If the first term of a progression containing an odd number of terms be a, and the last l, find the middle term (1) when the progression is arithmetical, and (2) when it is geometrical.

Sum the following series, each to n terms:—

$$a + a(1+r) + a(1+2r) + \ldots$$
$$a + a(1+r) + a(1+r)^2 + \ldots$$
$$1 + \frac{3}{2} + 2 + \ldots$$

7. If A vary as B when C is constant, and vary as C when B is constant, prove that A will vary as $B\,C$ when neither is constant.

Prove that if $a : b :: c : d$, then

$$\begin{aligned} a+b : a-b &:: c+d : c-d \\ &:: \sqrt{ac} + \sqrt{bd} : \sqrt{ac} - \sqrt{bd}. \end{aligned}$$

8. Extract the square root of

$$x^4 - 2x^3 + 3x^2 - 2x + 1,$$

and the fourth root of

$$x^2 + \frac{1}{x^2} + 4\left(x + \frac{1}{x}\right) + 6.$$

9. Three men, A B C, start at the same instant from the same place, P, to go to another place Q. A arrives first at Q, and immediately turns back and meets B at a point two-thirds of the distance from P to Q; B also then turns back to meet C, a hours after this meeting A meets C, and b hours after, B meets C. Find the time when A and B first met.

1862. *Wednesday, Jan.* 15*th.*—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. The price of 3 per cent. consols is $90\frac{3}{8}$, what sum must be invested in order to purchase £24 per annum? and what is the rate of interest on the money invested?

2. Three partners in trade contribute respectively the sums of £438, £292, and £730, with the agreement that each was to receive 5 per cent. on their respective investments, and that the remainder of the gains of the firm, if any, was to be divided between them in the proportion of the sums originally advanced. The whole gain of the firm was £200. What was each man's share?

3. Define a fraction. State and prove the rule for adding two fractions together.

Which of the two fractions $\frac{240}{193}$ and $\frac{1103}{887}$ is the greatest?

Simplify $\dfrac{1}{2+\dfrac{1}{3+\frac{1}{4}}}$ and $\dfrac{3\frac{1}{2}+2\frac{1}{4}}{3\frac{1}{2}-2\frac{1}{4}}$.

4. What is a decimal fraction? Show how to reduce any fraction to a decimal.

Express as decimals the following fractions:

$$\frac{2}{5}+\frac{3}{1000}+\frac{5}{100{,}000} \text{ and } \frac{1}{256}.$$

Find the value in shillings and pence of £0·94432, and express $6\frac{3}{4}d.$ as a fraction of a pound.

5. Find the value of $x^2 - 5x + 6$, first when $x = 2$, and secondly when $x = 3$.

Simplify the following expressions:

$$a + 3b - (5a - 9b) - (4a + 8b) - (-6a + b),$$

$$(8xy - 4xz)(2xy + xz) - 16x^2y^2,$$

$$\frac{\dfrac{x^4}{4} - x^2 + 1}{\dfrac{x^2}{2} - x + 1},$$

and write down the fifth power of $x - a$.

6. Solve the equations:

$$\frac{x-1}{3}+\frac{2x-3}{4}=\frac{5}{4}+\frac{2x+1}{9}. \qquad \left.\begin{aligned}\frac{1}{x}+\frac{2}{y}&=8\\ \frac{4}{x}-\frac{2}{y}&=2\end{aligned}\right\}.$$

7. Define proportion, and show that your definition agrees with that given by Euclid.

Prove that if $a : b :: c : d$, then $a + b : a - b :: c + d : c - d$. Find a third proportional to $a^3 + b^3 + a\,b\,(a + b)$ and $a^2 - b^2$.

8. Show how to sum n terms of a geometrical progression. Sum to infinity the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\dots$ and explain how an infinite number of terms can have a finite sum.

Sum the series:

$$\left.\begin{array}{l} 1 + \frac{3}{2} + 2 + \frac{5}{2} + \dots\dots \\ 1 - \frac{3}{2} + 2 - \frac{5}{2} + \dots\dots \end{array}\right\},$$

each to $2\,n$ terms.

9. A policeman runs after a boy who starts a little distance ahead. Three of the policeman's steps are equal to 5 of the lad's, but the boy takes 3 while the policeman takes 2 steps. Will the policeman catch the boy?

1862. *Wednesday, July 9th.*—Examiners,—W. H. BESANT, Esq., M.A., and E. J. ROUTH, Esq., M.A.

1. Define a fraction, and state the rules for the addition, subtraction, and multiplication of fractions.

Add together $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$, and divide the result by the sum of $\frac{1}{5}$, $\frac{1}{7}$, and $\frac{1}{9}$.

2. Find the value of $\frac{5}{17}$ of £3 12*s.* 11½*d.*, and find the fraction that 3 miles, 2 fur., 100 yds., is of 12 leagues, 2 miles, 2 fur., 20 yds.

3. If 25 tons of goods are purchased for £37 10*s.*, and sold at 35*s.* a ton, what is the gain per ton?

At what rate per ton should the goods have been sold, in order to obtain a profit of £9 7*s.* 6*d.*?

4. Find the simple interest on £5000 for 6 years, at 3½ per cent.

At what rate, simple interest, will £300 amount to £373 10*s.* in 7 years?

5. Show how to reduce a vulgar fraction to a decimal, and prove that it must be either a terminated or a recurring decimal.

Multiply ·00456 by ·000322, and find the square root of ·00207936.

Find the value of ·525 of £20, and also of $\cdot\dot{2}\dot{2}$ of £297.

6. Add together $x + y$, $3\,x - y - z$, and $4\,y - 2\,x - z$, and multiply the result by $x - y - z$.

Find the value of $\frac{x^5 - x^4 y + x^3 y^3}{y^5 - y^4 x + y^3 x^3}$, when $x = 2 y$.

And prove that $(x + y)(x^2 + y^2)(x^4 + y^4) = \frac{x^8 - y^8}{x - y}$.

Simplify the expression

$$\frac{1}{2} \cdot \frac{1}{x - 1} - \frac{x - 5}{x^2 - 7 x + 10} + \frac{1}{2} \cdot \frac{x - 6}{x^2 - 9 x + 18}.$$

7. If $a : b : : c : d$, prove that

$$a + b : a - b : : c + d : c - d,$$

$$\text{and } a^2 + c^2 : b^2 + d^2 : : \sqrt{a^4 + c^4} : \sqrt{b^4 + d^4}.$$

Determine whether the ratio of $a : b$ is increased or diminished by adding the same quantity to both its terms.

8. Define an arithmetic progression, and, having given the first term a and the common difference b, find the nth term, and the sum of n terms.

Sum the series

$$1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + \ldots\ldots \text{ to 20 terms,}$$

$$89 + 84\tfrac{1}{2} + 80 + \ldots\ldots \text{ to 18 terms;}$$

and find the sum of n terms of the series of which the rth term is $\frac{4 r + 5}{2}$.

9. Solve the equations,

$$\frac{x - 1}{5} - \frac{x - 11}{7} + \frac{3 x - (5 x - 4)}{2} + \frac{278}{35} = 0,$$

$$\left\{\begin{array}{l} 5 x + 11 y = 146 \\ 11 x + 5 y = 110 \end{array}\right\};$$

and find the value of x which makes the excess of $(2 x + a)^3$ over $(x + a)^3$ equal to $4 a (7 x^2 + 9 a x + 3 a^2)$.

10. On Monday, June 9th, the turnstiles recorded the entrance into the Exhibition of 58,682 persons. The money taken (in shillings) consisted of a number of pounds, and 7 shillings over, and it was observed, that the number of pounds was less, by 295, than the number of persons who entered with season-tickets. Find the number of persons who entered, respectively, by payment and by season-tickets.

1863. *Wednesday, Jan. 14th.*—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. A merchant sells tea to a tradesman at a profit of 60 per cent.; but the tradesman, becoming a bankrupt, pays only 2*s.* 6*d.* in the

pound. How much per cent. does the merchant gain or lose by his sale?

2. Multiply $99\frac{272}{333}$ by $1\frac{2}{3}$, and find the nearest integer to the product.

Show that $\frac{1}{2600}$ of £10 16*s.* 8*d.* is equal to ·002 of £2 1*s.* 8*d.*

Simplify

$$\frac{3 + \frac{5}{8} \text{ of } \frac{21}{7\frac{3}{5}} - \frac{1}{4} - \frac{1\frac{2}{3}}{2\frac{1}{2}}}{10 - \frac{151}{228} \text{ of } 5}.$$

3. Find the square root of ·006084. Divide ·03 by ·0000375, and find the value of ·53125 of £1.

4. Find the sum which must be invested in the 3 per cents. at 90, at simple interest payable half-yearly, to amount in $23\frac{1}{2}$ years to £2317 money, the price of the funds remaining unchanged. If the funds rose to 96, in how many years sooner could the required amount be realized?

5. Add together $s - a$, $s - b$, $s - c$, where $s = \frac{a + b + c}{2}$.

Subtract $\frac{2x^2 - 13x + 1}{x^2 - 1}$ from $\frac{5x - 3}{x + 1}$.

Write down the result of the division of $x^9 + y^9$ by $x + y$.

Simplify

$$\frac{\{(ax + by)^2 + (ay - bx)^2\}\,\{(ax + by)^2 - (ay + bx)^2\}}{x^4 - y^4}.$$

6. Solve the equations:

$$\frac{x+1}{3} - \frac{1}{2}\left(x + 3 - \frac{3}{2}x\right) = x - 2; \qquad \frac{2x+7}{x-1} - \frac{x+1}{x+7} = 1;$$

$$\frac{a}{x} + \frac{b}{y} = 1, \quad \frac{a'}{x} + \frac{b'}{y} = 1.$$

By reference to Bradshaw, it is found that two railway trains moving in opposite directions, pass each other somewhere between two stations *A* and *B*. One train leaves *A* at 20 minutes past 4, and arrives at *B* at 5. The other leaves *B* at 10 minutes past 4, and arrives at *A* at 20 minutes to 5. Find the time at which they meet.

7. Define proportion. Show that if $a : b :: c : d$, then $a\,d = b\,c$.

Prove also that if

$$\frac{a + b + f}{a - b - c} = \frac{a - b - c + d}{a + b + f - d} = \frac{c - d - f}{c - f + d},$$

each of the three ratios is equal to unity.

8. Find the first term of an arithmetical progression, having given the sum, the number of terms, and the common difference.

Sum the series

$$1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots\dots \text{ to infinity.}$$

$$1 - \frac{1}{4} - \frac{3}{2} - \frac{11}{4} - \dots\dots \text{ to 17 terms.}$$

If s be the sum of an odd number of integers in geometrical progression, and s' the sum of the squares of the same integers, prove that $\frac{s'}{s}$ is an integer.

9. *A* can dig a certain length of a ditch in 3 days, *B* can do the same in 4 days, and *C* in 5 days. How long will it take the three persons, *A*, *B*, *C*, together, to dig the same length of ditch?

What proportion of the length is dug by each?

1863. *Wednesday, July* 8*th*.—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Add together the fractions $\frac{1}{12}, \frac{1}{81}, \frac{1}{36}$; and subtract $\frac{1}{64}$ from the result.

Which is the greatest of the fractions $\frac{11}{12}, \frac{17}{18}, \frac{23}{24}$?

2. Find the value of $\frac{3}{10}$ of £2 + $\frac{4}{7}$ of half-a-guinea + 8*s*. 8*d*.

What fraction of a crown is the difference between $\frac{3}{4}$ of a shilling and $\frac{5}{8}$ of a guinea?

What fractions are 7*s*. 6*d*. of £1, and £1 2*s*. 6*d*. of £1 10*s*.?

3. If the wages of 6 men for 5 weeks be £6, how long will 8 men work for £10?

4. A grocer buys some tea at 4*s*. per lb., and some at 5*s*. 6*d*. In what proportion must he mix them, that when he sells the tea at 6*s*. per lb., he may gain 20 per cent.?

5. Divide ·0003 by ·005. Which is the greater, ·0262 of half-a-guinea, or ·38 of a crown? Reduce $\dfrac{3}{4 + \dfrac{2}{3 + \frac{2}{3}}}$ to a decimal fraction.

6. Add together $2x^2 - 3xy + y^2$, $x^2 + xy - y^2$, and $x^2 + 2xy$.

Find the product of the quantities

$$x + y - z, \qquad x - y + z, \qquad y + z - x;$$

and show that if y be put equal to nothing in the product, the result is $(x^2 - z^2)(z - x)$.

7. If $2A = B + C$, and $3B = C + A$, prove that $4B = 3A$.

Simplify $\dfrac{(x^2 + \sqrt{2} \,.\, x + 1)(x^2 + 1)(x^2 - 1)}{x^8 - 1}$.

If $a : b :: c : d$ prove that $b\,c = a\,d$, and

$$a^2 + a\,b + b^2 : c^2 + c\,d + d^2 :: a^2 + b^2 : c^2 + d^2.$$

8. Show how to sum an arithmetical progression when the first and last terms are given, and also the number of terms.

Sum the series

$$1 + \frac{3}{2} + 2 + \frac{5}{2} + \ldots\ldots \text{ to } n \text{ terms,}$$

$$\text{and } 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots\ldots \text{ to infinity.}$$

9. Solve the equations:

$$(1)\ \frac{x+1}{2} - \frac{2}{x+1} = \frac{x-1}{2}; \qquad (2)\ \left\{\begin{matrix} 2\,x + \ \ y = \ 8 \\ 3\,x + 5\,y = 19 \end{matrix}\right\}.$$

10. It is required to divide 36 into three such parts, that one-half of the first, one-third of the second, and one fourth of the third, may be equal to each other.

1864. *Wednesday, January* 13*th*.

Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Find the simple interest of £1,125 for 4 years, at $1\frac{1}{2}$ per cent.

Distinguish between discount and interest. If the discount on a certain sum of money due 15 months hence be £6 4*s.*, find the amount, the rate being 5 per cent.

2. Two persons buy apples at a penny each; one sells them at five for 6*d.*, and the other at six for 7*d.*: compare the gains per cent. in the two cases.

3. Find the value of $\frac{5}{7}$ of $\frac{4}{9}$ of $1\frac{3}{11}$ of $6\frac{1}{20}$; and reduce $\frac{19325}{22005}$ to its lowest terms.

What fraction is $\frac{1}{3} \times 3\frac{1}{2}$ of $\dfrac{\frac{\frac{2}{35}}{5}}{7\frac{1}{3}}$?

Simplify $\dfrac{\frac{1}{2}\left\{\frac{1}{3} - \frac{1}{14} + \frac{6}{7}\left(\frac{1}{2} - \frac{1}{3}\right)\right\}}{1 + \frac{1}{2 + \frac{1}{3}}}$.

Show that if the numerator and denominator of a fraction be multiplied by the same number, the value of the fraction is not altered.

4. Divide ·0022428 by 2·67, and reduce $\dfrac{3^2}{2^3 \times 5^2}$ to a decimal.

Find the value of ·0625 of £21 6*s.* 8*d.*; and express 1*s.* $1\frac{1}{2}$*d.* as a fraction of 10*s.*

Extract the square root of ·0029929.

5. Define the greatest common measure (1) of two arithmetical numbers, and (2) of two algebraic expressions.

Find the *g. c. m.* of $x^3 + 4x^2 + 2x - 1$ and $x^3 + 2x^2 - 4x + 1$.

Simplify $\dfrac{x}{(x-y)(x-z)} + \dfrac{y}{(y-z)(y-x)} + \dfrac{z}{(z-x)(z-y)}$; and divide $x^4 + 2x^3 - 2x - 1$ by $(x-1)^2$.

6. Solve the equation, $\dfrac{x-1}{x+2} + \dfrac{x-3}{x-2} = 2$.

If a regiment of soldiers be arranged in a solid square, there are eleven men over; but there are too few by fifty to form a square with one more on each side. How many men are there in the regiment?

7. Prove that if four quantities be proportional, the first is to the third as the second is to the fourth.

If $a : b :: c : d$, prove that

$$\frac{a^2 + ab + b^2}{a^2 - ab + b^2} = \frac{c^2 + cd + d^2}{c^2 - cd + d^2}.$$

8. Given the first and last terms of an arithmetical progression and the number of terms, find the middle term or the two middle terms of the series. Find also the sum of the series.

Sum the series

$$\tfrac{1}{3} + \tfrac{2}{3} + 1 + 1\tfrac{1}{3} + \ldots\ldots\ldots \text{ to } n \text{ terms;}$$

$$\tfrac{1}{3} + \tfrac{2}{3} + \tfrac{4}{3} + \ldots\ldots\ldots\ldots \text{ to } n \text{ terms.}$$

9. A tax having been increased from 5 per cent. to 7 per cent., the revenue derived from it was found to be diminished by one-third. Find the diminution in the consumption of the article.

1864. *Wednesday, June* 29.—Examiners,—W. H. Besant, Esq., M.A., and Isaac Todhunter, Esq., M.A., F.R.S.

1. Define multiplication; and multiply 4567 by 567, explaining the process.

In a manufactory there are 273 workmen; of whom 100 are paid each 12*s.* 6*d.* a week, 100 are paid 15*s.* 4*d.* a week, and the remainder 16*s.* 8*d.* Find the sum expended weekly in wages.

2. Add together the fractions $\frac{1}{15}$, $\frac{5}{24}$, $\frac{11}{36}$, and $\frac{7}{40}$; and divide the result by $\frac{11}{25}$ of $7\frac{11}{12}$.

Find the value of $\frac{5}{6}$ of £75 16*s.* 8*d.* + $\frac{1}{3}$ of £2 10*s.* 2*d.*; and determine what fraction £10 2*s.* 6*d.* is of £151 17*s.* 6*d.*

3. Reduce $\frac{117}{625}$ to a decimal; and divide ·00042 by ·000625.

Find the square root of 31990336 and of ·047961; and determine the value of ·625 of £51·2.

4 Distinguish between simple and compound interest.

Find the simple interest on £45,732 17*s.* 8*d.* for $3\frac{1}{4}$ years, at 4 per cent. per annum.

Which is the better investment, bank stock paying 10 per cent. at 317, or 3 per cent. consols at 95?

5. If $b > c$, and $a > b - c$, prove that $a - (b - c) = a - b + c$.

Add together $x^3 + 4x^2y - 4y^3$, $3x^3 - 5xy^2$, and $5y^3 - 3x^3 - x^2y - xy^2$; and subtract the result from $(x + y)^3$.

Multiply together $x^2 - xy + y^2$, $x^2 + xy + y^2$, and $x^4 + x^2y^2 + y^4$; and find the value of the product when $x = 1$ and $y = 2$.

6. Prove that $\dfrac{x^n - y^n}{x - y} = x^{n-1} + y\,\dfrac{x^{n-1} - y^{n-1}}{x - y}$; and that if n be a positive integer, $x^n - y^n$ is always divisible by $x - y$.

Divide $x^4 + 4x^3 + 8x^2 + 41x + 36$ by $x^2 - x + 9$; and simplify the expression

$$\frac{x}{(x - y)(x - z)} + \frac{y}{(y - z)(y - x)} + \frac{z}{(z - x)(z - y)}.$$

7. Define proportion; and find a fourth proportional to the quantities a^2bc, $a^3b^2c^3$, and a^5bc^6.

If $a : b :: c : d$, prove that $a + b : a - b :: c + d : c - d$; and that $a^3 + 3a^2b + b^3 : a^3 - 3ab^2 + b^3 :: c^3 + 3c^2d + d^3 : c^3 - 3cd^2 + d^3$.

8. Having given the first term and the common difference, find the sum of n terms of an arithmetical progression.

Find the sums of the arithmetical series,

$4 + \frac{13}{3} + \frac{14}{3} + 5 + \ldots\ldots\ldots$ to 100 terms;
and $24 + 23\frac{1}{2} + 23 + 22\frac{1}{2} + \ldots\ldots$ to 50 terms.

Also find the rth term, and the sums to n terms and to infinity of the geometric series

$$1 + \tfrac{2}{3} + \tfrac{4}{9} + \ldots\ldots$$

9. Solve the equations,

$$(1)\quad \frac{x + 1}{15} - \frac{x - 9}{16} = \frac{x + 9}{5} - \frac{x + 1}{4};$$

$$(2)\quad (x + a)(x + b) = (x + 2a)(x + 2b);$$

$$(3)\quad \begin{cases} 15x + 16y = 108, \\ 4x + 5y = 31. \end{cases}$$

10. At the late volunteer review in Hyde-park it was observed that the total number of men, if increased by 497, would have formed sixteen brigades, each equal in strength to the Prince of

Wales's brigade; and that if the Prince of Wales's brigade had been stronger by 1000 men, the total number would have exceeded nine of such brigades by 233. Find the total number, and also the number of men in the Prince's brigade.

1865. *Wednesday, January* 11*th*.

Examiners,—W. H. Besant, Esq., M.A., and J. Todhunter, Esq., M.A.

1. A bankrupt's estate amounts to £910 3*s*. $1\frac{1}{2}$*d*., and his debts to £1875. What can he pay in the pound? and what will a creditor lose on a debt of £57?

2. Add together $\frac{5}{12}$, $1\frac{3}{8}$, $\frac{1\frac{1}{2}}{8}$, and $\frac{1\frac{3}{4}}{1\frac{5}{8}}$; and subtract the result from $4\frac{1}{4}$.

Find the value of $\frac{\cdot 003 \times \cdot 004}{\cdot 006}$, and of $\frac{\cdot 10724}{\cdot 003125}$.

3. Find the square root of ·05368489.

Find the value of $\frac{3}{\sqrt{8} - 1}$ to four places of decimals.

4. A person having invested a sum of money in the 3 per cent. consols, receives annually therefrom £233; after deducting the income tax of 7*d*. in the pound, what is the sum of money? What can the stock be sold for when the consols are at $94\frac{1}{8}$?

5. Simplify $(x + y)^3 + (x + y)^2 y + (x + y) y^2 - \{3 x^2 y + 5 y^2 x + 2 y^3\}$ and $\frac{a^2 - 3 a b + 2 b^2}{a - 2 b} - \frac{a^2 - 7 a b + 12 b^2}{a - 3 b}$.

6. Solve the following equations:

(1) $\frac{1}{3}(5 x - 2) - \frac{1}{7}(21 - 5 x) = 41 - 4 x$;

(2) $\frac{6 x + 21}{11} - \frac{8 x - 2}{7} = \frac{4 x - 3}{5}$;

(3) $\frac{12}{x} + \frac{8}{y} = 5$; $\frac{27}{x} - \frac{12}{y} = 6$.

7. A person bought some yards of cloth for 120*s*. If there had been 6 yds. more, each yard would have cost a shilling less. Required the number of yards, and the price of each.

8. Define proportion.

Show that if $a : b :: c : d$, then

$$\frac{a}{c} = \frac{a + b}{c + d},$$

and $\frac{p a^2 + q a b + r b^2}{l a^2 + m a b + n b^2} = \frac{p c^2 + q c d + r d^2}{l c^2 + m c d + n d^2}$.

9. Show how to find the sum of a given number of terms in

an arithmetical progression, the first term and the common difference being supposed known.

The first term of an arithmetical progression is 11, and the fourth term 17. Find the sum of ten terms.

The second term of a geometrical progression is 3, and the fifth term 24. Find the sum of eight terms.

1865. *Wednesday, June 28th.*—Examiners,—E. J. Routh, Esq., M.A., and Isaac Todhunter, Esq., M.A., F.R.S.

1. Reduce the expressions

$$\tfrac{3}{20} + \tfrac{5}{12} + \tfrac{4}{15} - \tfrac{5}{6}, \text{ and } \left\{\frac{\frac{1}{4} + \frac{1}{5}}{\frac{1}{4} - \frac{1}{5}} + \frac{\frac{1}{3} + \frac{1}{4}}{\frac{1}{3} - \frac{1}{4}}\right\}^2.$$

Multiply $49\tfrac{15}{16}$ by $50\tfrac{1}{16}$, and add $\tfrac{1}{256}$ to the result.

Divide $(2\tfrac{1}{16})^3 - 1$ by $(2\tfrac{1}{16})^2 + 3\tfrac{1}{16}$.

2. Divide ·28413 by ·0231.

Extract the square root of 1095·61, and find to three places of decimals the value of $\dfrac{4}{\sqrt{5} - 1}$.

What fraction of a crown is $\tfrac{3}{5}$ of 6*s*. 8*d*.? What is the value of $\tfrac{2}{7}$ of a guinea? Reduce $11\tfrac{3}{4}$*d*. to a decimal of a pound, correct to five places of decimals.

3. Find the compound interest of £55 for one year, payable quarterly, at 5 per cent. per annum.

A person bought into the three per cents. at 98, and after receiving three years' interest, he sold at 90. How much per cent. on the sum invested did he gain or lose?

4. Three gardeners working all day can plant a field in 10 days; but one of them having other employment can only work half-time. How long will it take them to complete the work?

5. Define the greatest common measure of two numbers, and also of two algebraical quantities.

Find the greatest common measure of $x^3 - 1$ and $x^3 + 2x^2 + 2x + 1$.

Add together $\dfrac{a^2 + b^2}{2}$ and $ab + bc + ca$, where $a = -(b + c)$. Write down the fourth power of $a - b$.

Simplify $\dfrac{1}{x + 5} - \dfrac{3}{x + 3 - \dfrac{6}{x + 2}}$

and $\dfrac{1}{(x + 1)(x + 2)} + \dfrac{1}{(x + 2)(x + 3)} + \dfrac{1}{(x + 3)(x + 4)}$.

6. Solve the equations,

$$\frac{x+3}{4} - \frac{x-1}{2} = \frac{1}{3}\left(x + \frac{x}{2} + \frac{3}{2}\right);$$

$$\left.\begin{aligned} 3x - y &= 3 \\ 5x + 7y &= 31 \end{aligned}\right\};\qquad \left.\begin{aligned} x + y &= a \\ y + z &= b \\ z + x &= c \end{aligned}\right\}.$$

7. Given in an arithmetical series of an odd number of terms the middle term and number of terms; find the sum of the series.

Sum the series (each to n terms)

$$1 + 3 + 5 + 7 + \ldots\ldots\ldots$$

$$1 + 3 + 9 + 27 + \ldots\ldots\ldots$$

Find also the sum of the squares of the n terms of the latter series.

8. Prove that if $a : b :: c : d$, then

$$ma + nb : pa + qb :: mc + nd : pc + qd.$$

Prove that if $\dfrac{a^2 + ab + b^2}{c^2 + cd + d^2} = \dfrac{a^3 + b^3}{c^3 + d^3} \times \dfrac{d}{b}$, then $a : b :: c : d$.

9. There is a certain number of two digits which is 7 times the sum of the digits. If the number be read backwards, and subtracted from the original number, the difference is 27. Find the number.

PASS EXAMINATION PAPERS,

FOR

THE DEGREE OF BACHELOR OF ARTS.

1839. *Monday, May 27th.*—Examiner,—Mr. JERRARD.

1. IN what does the peculiar excellence of the present system of numerical notation consist?

2. Find the value of $\frac{355}{113}$ to six places of decimals, and of $\frac{0{\cdot}01331}{1{\cdot}3173}$ to four places.

3. Solve the following equations:

$$(1)\quad 2x - \frac{1}{2}(x+3) = 6.$$

$$(2)\quad (x+2)(x+3) = x(x+4).$$

$$(3)\quad x^2 - 4x + 3 = 0.$$

$$(4)\quad \sqrt{x^2 - 2x + 93} - \frac{x^2}{2} = 45 - x.$$

4. Prove that $\cos. 2\theta = 1 - 2(\sin. \theta)^2$.

5. In every plane triangle the sides are as the sines of the opposite angles.

6. Find the equation to the ellipse referred to rectangular axes, the origin being at the centre.

(*Wednesday, May 29th.*)

7. Find the area of a rectangular court, of which the diagonal is 100 yards, and the breadth 42 yards.

8. Solve the following equations:

$$(1)\quad \sqrt{x} + \sqrt{(10+x)} = \frac{20}{\sqrt{(10+x)}}.$$

$$(2)\quad \begin{cases} x^3 + y^3 = 1001, \\ x + y = \quad 11. \end{cases}$$

9. Given two sides and an included angle of a triangle, express the third side in a form adapted to logarithmic computation, the tabular radius being 10^{10}.

10. Find the equation to an ellipse referred to the axis minor and the tangent at its extremity as axes.

1840. *Monday, May 24th.*—Examiner,—Mr. JERRARD.

1. What extension takes place in the meaning of the term Multiplication, when applied to fractions? Investigate the general rule for multiplying any number of fractions together, and find the product of the decimals ·0101 and ·0202.

2. Explain the ordinary method of extracting the square root of a number. What is the square root of 2085·7489?

3. Find the simple interest of £352 11*s.* 2*d.* for 21 years and 3 months, at $3\frac{1}{4}$ per cent. per annum.

4. Solve the following equations:

$$(1)\quad \frac{5x-7}{2} - \frac{2x+7}{3} = 3x - 14.$$

$$(2)\quad \frac{x-1}{x-2} = \frac{x-3}{x-4} + 1.$$

$$(3)\quad 2x + 3y = 37,\quad \frac{1}{x} + \frac{1}{y} = \frac{14}{45}.$$

There are three pipes, one of which fills a reservoir in 4 hours, another in $3\frac{1}{2}$ hours, and the third in $2\frac{1}{4}$ hours; what time will they take to fill it when they are all flowing together?

5. Sum the series

$$(1)\quad \frac{1}{3} + \frac{2}{3} + 1 + \ldots \text{ to 50 terms.}$$

$$(2)\quad \frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \ldots \textit{ ad infinitum.}$$

6. Find the number of combinations that may be formed with m quantities, taken n at a time.

7. Show that $\log. (a\,b\,c\,d \ldots) = \log. a + \log. b + \log. c + \log. d + \ldots$

What are the advantages of Briggs's system of logarithms?

8. Define the sine and cosine of an angle. Also prove that

$$(1)\quad (\sin. A)^2 + (\cos. A)^2 = 1.$$

$$(2)\quad \cos. A = \frac{1}{\sqrt{1 + (\tan. A)^2}}.$$

9. Investigate the general formula

$$\text{Sin. } (A - B) = \sin. A \,.\, \cos. B - \cos. A \sin. B,$$

and show how it is verified when

$$\text{I. } B = 0. \quad \text{II. } A = B. \quad \text{III. } A = \frac{\pi}{2}.$$

10. Express the area of a triangle in terms of its sides.

(*May 27th.*—Mr. MURPHY.)

11. Find the equation to a circle, when the origin is a point in

the circumference, and the axis of x a diameter passing through that point. Considering the ellipse as the Orthographic Projection of a circle, deduce its equation from that of the circle.

1841. *Monday, May* 31*st.*—Examiner,—Mr. Jerrard.

1. State the nature of the questions to which the *Rule of Three* is applicable; and show that a rule similar in principle may be applied to questions involving more than three quantities.

How many men can complete a trench of 468 yards in 8 days, if 24 men can dig 81 yards in 6 days?

2. Find the value of $\frac{\sqrt{\cdot 012}}{\sqrt{2} + \sqrt{3}}$ to 4 places of decimals.

3. Given log. 3 $=$ ·4771213, log. 7 $=$ ·8450980, find the logarithm of 1323 and that of 1·323. 1323 is equal to $3^3 \times 7^2$. What is the criterion of the divisibility of a number by 3?

4. In what time will a sum of money double itself at 3½ per cent. per annum compound interest?

$$\text{Log. } 2 = \cdot 3010300,$$
$$\text{log. } 1\cdot 035 = \cdot 0149403.$$

5. Determine by actual multiplication the expansion of $(1 - 2b)^5$. Divide $a - b^2$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$. Also find the series which results from the division of 1 by $1 + x$.

What is the most simple form to which $\frac{3x^2 + 6x + 3}{2x^3 + 2}$ can be reduced?

6. Solve the equations:

$$(1)\quad \frac{x}{2} + \frac{x}{3} - \frac{x}{4} = 7. \qquad (2)\quad \frac{1}{1+x} + \frac{1}{1-x} = \frac{8}{3}.$$

$$(3)\quad \left.\begin{array}{l} 2x + 5y = 89, \\ 7x - 8y = 31. \end{array}\right\} \qquad (4)\quad \left.\begin{array}{l} xy = x - y, \\ x + y = 1. \end{array}\right\}.$$

$$(5)\quad x^2 - 3x - 130 = 0.$$

$$(6)\quad 7\sqrt{2x^2 - 10x + 3} = 72 + 5x - x^2.$$

7. How many variations can be made of the letters in the word *language?*

8. The sum of a decreasing arithmetic series is 140, the first term 10, and the common difference ⅓; find the number of terms.

Find the sum of 15 terms of the series $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$

9. Assuming the expression for sin. $(\alpha + \beta)$, and that for cos. $(\alpha + \beta)$, prove that

$$(1)\quad \tan.(\alpha + \beta) = \frac{\tan.\alpha + \tan.\beta}{1 - \tan.\alpha \tan.\beta},$$

and apply it to deduce the equation

$$(2)\quad \tan.\,(45^\circ + \beta) = 2\tan.\,2\beta + \tan.\,(45^\circ - \beta),$$

for determining the tangents of angles greater than 45°.

10. Show that in any plane triangle $c^2 = a^2 + b^2 - 2\,a\,b\cos.\,C$, where a, b, c, are the sides, and C is the angle opposite to c. How may this expression be adapted to logarithmic calculation?

(*June 2nd.*—Rev. R. Murphy.)

11. Find the equation to an ellipse referred to its axes major and minor.

1842. *Monday, Oct. 3rd.*—Examiner,—Mr. Jerrard.

1. Explain fully what is meant by a fraction, and show that the value of a fraction is not altered by multiplying the numerator and denominator by the same number. What decimal of £1 is $\frac{3}{8}$ of a guinea?

2. A person travelled from Slough to London, a distance of 18¼ miles, in twenty-five minutes; at what rate is this per hour?

How much would 456 tons cost at £3 17*s.* 5¾*d.* per ton?

3. Find the continued product of

$$x - a,\ x + a,\ x - a\sqrt{-1},\ x + a\sqrt{-1},$$

and reduce $\dfrac{x^3 - 6\,x^2 - 37\,x + 210}{x^3 + 4\,x^2 - 47\,x - 210}$ to its most simple form. What is the value of this fraction, when $x = 7$?

4. Solve the equations,

$$(1)\quad \frac{x}{7} - \frac{x-5}{11} + 5 = x - \left(\frac{2\,x}{77} + 1\right).$$

$$(2)\quad \left.\begin{array}{l} 3\,x + 11\,y = 84 \\ 7\,x - 19\,y = 62 \end{array}\right\}. \qquad (3)\quad (5\,x - 3)^2 - 7 = 44\,x + 5.$$

$$(4)\quad \sqrt{x - 8} = 2 + \sqrt{\tfrac{1}{2}\,x}. \qquad (5)\quad \left.\begin{array}{l} x + y = 18 \\ x^3 + y^3 = 4914 \end{array}\right\}.$$

When will the hour and minute hands be at right angles to each other between twelve and one o'clock?

5. Show that $\log.\,\frac{a}{b} = \log.\,a - \log.\,b$, $\log.\,a^m = m\log.\,a$.

Given $\log.\,2 = \cdot 3010300$, find the logarithm of 5, and that of 2000.

6. Define the sine and cosine of an angle, and show that

$$\text{Sin.}\,(A + B) = \sin.\,A\cos.\,B + \cos.\,A\sin.\,B,$$

where A and B are any two angles.

7. In any triangle of which A, B, are two angles, and a, b, the opposite sides, prove that $\frac{a}{b} = \frac{\sin.\,A}{\sin.\,B}$. What will be the ratio of a to b, if $A = 45^\circ$, $B = 30^\circ$? Given a, b, A, discuss the ambiguity which may arise in determining the triangle.

8. Express, in a form adapted to calculation by logarithms, the area of a triangle in terms of the sides. What is the area of the triangle of which the sides are 3, 5, and 7 feet in length?

(*Oct. 5th.*—Rev. R. MURPHY.)

9. Find generally the equation to the common parabola referred to rectangular co-ordinates, the origin being arbitrary.

1843. *Monday, Oct. 2nd.*—Examiner,—Mr. JERRARD.

1. Reduce $\frac{1}{17}$ to a decimal, and extract the square root of ·00007038409.

2. Explain the nature and use of logarithms, and point out the advantages of the common system over that of Napier.

3. Prove the truth of the rule for the signs in multiplication. Find the fifth power of $(2a - 3b)$, and multiply $x^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$.

4. Investigate the rule for finding the greatest common measure of two algebraic quantities, and apply it to reduce

$$\frac{x^3 - 17x^2 + 79x - 63}{x^3 - 13x^2 + 15x + 189}$$

to its lowest terms.

5. Find the expression for the sum of n terms of the series,

$$a + ar + ar^2 + \ldots,$$

and sum the series,

$$2 + 6 + 18 + \ldots \text{ to 12 terms,}$$

$$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots \textit{ ad infinitum,}$$

$$1 + 1\tfrac{1}{2} + 2 + \ldots \text{ to 29 terms.}$$

6. Solve the equations,

$$(1) \quad \frac{7x + 5}{4} - \frac{x - 8}{3} = \frac{11x}{5}.$$

$$(2) \quad \left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\}. \qquad (3) \quad \left.\begin{aligned} x + y &= 21 \\ xy &= 61 \end{aligned}\right\}.$$

$$(4) \quad x^2 - 3x + 5 = 7\sqrt{x^2 - 5x + 6} + 2x + 17.$$

Required three numbers in geometric progression whose sum is 91, and the sum of their squares 4459.

7. How many degrees, minutes, &c., in the centesimal division of the quadrant, correspond to 101° 2′ 34″ in the nonagesimal system?

8. Given the sines and cosines of two angles, find the sine and cosine of (1) the sum and (2) the difference of the angles.

(*Oct. 5th.*—Rev. Prof. HEAVISIDE.)

9. Find the general equation to a circle referred to rectangular co-ordinates. How does the equation become modified by taking the origin, (1) in the circumference, (2) at the centre of the circle?

1844. *Monday, Oct. 7th.*—Examiner,—Mr. JERRARD.

1. Reduce $\frac{13}{625}$ and $\frac{5}{11\frac{1}{2}}$ to decimal fractions. Divide 53·796 by 7·82, and explain the process.

2. If 17 cwt. 3 qrs. 11 lbs. cost £256 15*s.* 7*d.*, what will 3 cwt. 1 qr. 21 lbs. cost?

3. Expand $(a + 2\,b - 3\,c)^3$. Divide $a^2 - b^2$ by $a + b$. Also find the quotient of $\frac{a^2 + b^2}{a + b}$ to five terms. What is the meaning of such a result as $(a + b\sqrt{-1})\,(a - b\sqrt{-1}) = a^2 + b^2$?

4. Required the number of permutations of n things taken m and m together.

5. Show how to find the sum of any number of terms (1) of an arithmetic, (2) of a geometric series.

Examples. $1 + 5 + 9 + \ldots$ to 32 terms.

$\frac{1}{5} + \frac{1}{3} + \frac{5}{9} + \ldots$ to 17 terms.

6. Find the value of x in the equation,

$$\frac{5\,x + 7}{9} = \frac{4\,x}{13} - \frac{x - 10}{3} + 5.$$

Explain the method of solving the quadratic equation,

$$x^2 + A_1\,x + A_2 = 0,$$

and show that, if its roots be denoted by x, and x_2, we shall have

$$x_1 + x_2 = -A_1,\; x_2 = A_2.$$

Take as an example, $x^2 + 6\,x - 55 = 0$.

Also find x and y from the equations,

$$x^{x+y} = y^{4a},\quad y^{x+y} = x^a,$$

by means of logarithms.

7. Define the sine, cosine, secant, tangent, and versed sine of an angle, and prove that $\tan.\,(A + B) = \frac{\tan.\,A + \tan.\,B}{1 - \tan.\,A\,\tan.\,B}$. What does this expression become, when $A = 45°$, $B = 30°$?

8. Express the sine of an angle of a triangle in terms of the sides, and in a form adapted to logarithms. Give a general account of the method of observing angles; and show how to find the height and distance of an inaccessible object on a horizontal plane.

(*Oct.* 9*th.*—Rev. Prof. HEAVISIDE.)

9. Find the equation to the ellipse referred to rectangular co-ordinates, the origin being taken either at the centre, or at the extremity of its major axis.

1845. *Monday, Oct. 6th.*—Examiner,—Mr. Jerrard.

1. Express $\frac{3}{4}$ as a decimal. What is the form of fractions which are convertible into finite decimals? Show that the recurring periods for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots \frac{6}{7}$, will consist of the same digits.

2. Two persons have gained in trade £375; the one put in £500, and the other £850: what is each person's just share of the profit? What will the rates of a parish of which the rental is £2154 15*s.* 6*d.* amount to, at 7½*d.* per pound?

3. Explain the rule of signs in the multiplication of algebraic quantities. Expand $(x + a)(x + b)(x + c)$.

What will this expression become, when

$$(1)\quad a = b = c. \qquad (2)\quad b = -a,\ c = 0?$$

Also divide $x^3 - 4x^2 - 7x + 10$ by $x^2 + x - 2$.

4. Place three arithmetic means between 1 and 7.

Solve the equations:

$$(1)\quad \frac{1}{4} - (4x + 3) = \frac{x}{3} - 6. \qquad (2)\quad \left.\begin{array}{r} 5x + 6y = 89 \\ 12x - 7y = 21 \end{array}\right\}.$$

$$(3)\quad x^2 + 14x - 51 = 0;$$

and explain the process for the solution of quadratic equations.

5. Show that the number of permutations of n things, taken r and r together, is

$$n(n - 1)(n - 2) \ldots (n - r + 1).$$

How many distinct permutations can be formed with the letters in the word *degree?* What is the general expression for the number of permutations, when there are different sets or classes of identical letters?

6. Define the term *logarithm.* What is meant by the *mantissa?* Given

$$\text{Log. } 2 = \cdot 3010300,$$
$$\text{log. } 3 = \cdot 4771213,$$

to find log. 45, log. 450, log. 4·5. State the advantages of the base of the system of logarithms being coincident with the base of our system of arithmetical notation. Why in every system will the logarithm of 1 be 0?

7. Define the sine and cosine of an angle, and trace their changes in sign through four right angles. Also show that

$$\text{Sin. } (A \pm B) = \text{sin. } A \text{ cos. } B \pm \text{cos. } A \text{ sin. } B.$$

8. Express the area of a triangle in terms of its sides.

(*Oct. 8th.*—Rev. Prof. Heaviside.)

9. Find the rectangular equations—

(1) To a straight line passing through a given point, at right angles to a given straight line.

(2) To the parabola.

1846. *Monday, Oct. 26th.*—Examiner,—Mr. Jerrard.

1. Find the difference between $\frac{355}{113}$ and $\frac{157}{50}$, and express $\frac{1}{13}$ as a decimal. What is the test of arithmetical equality?

2. What will £650 amount to in 5 years, at 5 per cent., compound interest?

3. Extract the square root of 9622404, and that of 96224·04; and explain the process.

4. "A ship's company take a prize of £1000, which is to be divided amongst them according to their pay, and to the time during which they have served: now the officers, four in number, have 40*s.* each a month; the midshipmen, 12 in number, have 30*s.* each a month; and they have all served 6 months: the sailors, who are 110 in number, have each 22*s.* a month, and have been on board 3 months. What will be the share of each class?"

5. What is meant by the product of two fractions? When are four quantities said to be proportionals? If $a, b, c, d,$ be proportionals, prove that,

$$(1)\quad a + b : a - b :: c + d : c - d.$$
$$(2)\quad a^n : b^n :: c^n : d^n.$$

6. Investigate an expression for the sum of n terms, (1) of an arithmetic, (2) of a geometric series: and solve the equations,

$$(\alpha)\quad \frac{7x + 2}{3} - \left(x - \frac{x - 6}{5}\right) = 1.$$

$$(\beta)\quad \left.\begin{aligned} 2x + 3y &= 47 \\ 7x - 2y &= 27 \end{aligned}\right\}. \qquad (\gamma)\quad \left.\begin{aligned} xy &= 105 \\ x^2 + y^2 &= 274 \end{aligned}\right\}.$$

Also form an equation the roots of which shall be 2 and 3.

7. Show that the number of combinations of n things, taken r and r together, is equal to the number of combinations of n things taken $n - r$ and $n - r$ together.

8. Assuming the expression for sin. $(A \pm B)$ and cos. $(A \pm B)$, prove that

$$\text{Tan.}\ (A \pm B) = \frac{\tan. A \pm \tan. B}{1 \mp \tan. A \tan. B}.$$

9. Find an expression adapted to logarithmic computation for the area of a triangle in terms of its sides.

(*Oct. 28th.*—Rev. Prof. Heaviside.)

10. If $\frac{x}{a} + \frac{y}{b} = 1$ be the equation to a straight line referred to

rectangular co-ordinates, what do a and b represent geometrically? Find the equation to the line drawn from the origin perpendicular to the above line.

11. Find the rectangular equation to the ellipse, the centre being the origin.

1847. *Monday, Oct. 25th.*—Examiner,—Mr. JERRARD.

1. What is the present value of £2063 14*s.* due 6 months hence, interest being at 3 per cent.?

2. Extract the square root of 898893·61.

3. Divide $a^n - b^n$ by $a - b$; and reduce

$$\frac{x}{1-x} - \frac{x^2}{1-x^2} - \frac{x^3}{1-x^3}$$

to its most simple equivalent form. What meaning do you assign to a^n, when n is negative or fractional?

4. Give the algebraical definition of proportion, and deduce Euclid's definition for it. Also show that if $a : b :: c : d$, we shall have $a^2 + b^2 : a^2 - b^2 :: c^2 + d^2 : c^2 - d^2$.

5. In how many different ways can seven persons be arranged on seven seats?

6. Solve the equations:

$$(\alpha) \quad \frac{x}{2} - (7x - 40) = \frac{12 - 2x}{11} + 1.$$

$$(\beta) \quad \left.\begin{aligned} 11x + 7y &= 47 \\ 23x - 29y &= 11 \end{aligned}\right\}. \qquad (\gamma) \quad \left.\begin{aligned} x^2 + y^2 &= 146 \\ x + y &= 16 \end{aligned}\right\}.$$

$$(\delta) \quad x^2 - 12x + 32 = 0.$$

7. Show that the logarithm of the pth power of a number is p times the logarithm of the number.

8. Given the sines and cosines of two angles, find the sine and cosine of their sum and difference; and show that

$$\text{Sin. } 36^\circ = \frac{\sqrt{5 - \sqrt{5}}}{2\sqrt{2}}, \text{ cos. } 36^\circ - \frac{\sqrt{5} + 1}{4}.$$

(*Oct. 27th.*—Rev. Prof. HEAVISIDE.)

9. In the equation to a straight line $y = ax + b$, what do a and b represent geometrically? Determine (a) and (b) when the straight line passes through two points whose co-ordinates are (x', y') $\left(0, \frac{y'}{2}\right)$.

10. Find the rectangular equation to the parabola, the vertex being the origin.

1848. *Monday, Oct. 23rd.*—Examiner,—Mr. JERRARD.

1. Show that when a number is divisible by 4, its two last figures must be divisible by 4; when by 9, the sum of its digits is divisible by 9; and resolve 1679616 into its factors.

2. Explain the method of extracting the square root of a number. Take as an example 1679616. What is the square root of $2\frac{3}{4}$ to five places of decimals?

3. Find the simple interest of £238 10*s.* 10*d.* for 3 years, at $4\frac{1}{2}$ per cent.

4. Multiply $a + \frac{x}{2a} - \frac{x^2}{4a^2}$ by $x - \frac{a}{2x} + \frac{a^2}{4x^2}$, and reduce $\frac{a+b}{a-b} + \frac{a-b}{a+b}$ to its most simple equivalent form. Explain the rule of signs in the multiplication of algebraic quantities.

5. Solve the equations,

$$(\alpha)\quad \frac{8x}{5} - \left(\frac{x}{7} - 30\right) - 2x = 11. \qquad (\beta)\quad \left.\begin{array}{l} 4x + 5y = 73 \\ 4y - 3x = 15 \end{array}\right\}.$$

$$(\gamma)\quad x^4 + qx^2 + s = 0.$$

The sum of a decreasing arithmetical series is 75, its first term 21, and the common difference 3; to find the number of its terms.

6. Show that in a system of logarithms to the base 10, the logarithms of all numbers which are expressed by the same succession of significant digits, may be found from one opening of the tables.

7. Define the sine and cosine of an angle, and trace their variations through four right angles. Also prove that in a triangle the sides are proportional to the sines of the opposite angles.

8. To express the area of a triangle in terms of the sides.

(*Oct. 25th.*—Rev. Prof. HEAVISIDE.)

9. Find the general equation in rectangular co-ordinates to a given circle. And from this deduce the equations (1) when the origin is in the circumference, (2) when it is at the centre of the circle.

1849. *Monday, Oct. 22nd.*—Examiner,—Mr. JERRARD.

1. What is the meaning of the term multiplication, as applied to fractions? State the general rule for multiplying any number of fractions together. Also reduce $\frac{36}{1250}$ to a decimal, and divide ·00031 by ·32.

2. A privateer running at the rate of 10 miles an hour, discovers a ship 18 miles off, making way at the rate of 8 miles

an hour: how many miles can the ship run before she will be over taken?

3. Show that in a system of logarithms to the base 10, the logarithms of all numbers between 1 and 10 must be included between 0 and 1. How is the characteristic of a logarithm generally determined? Given log. 2 = ·3010300, log. 3 = ·4771213, to find log. 80, log. 81, log. 360. Point out the utility of logarithms in calculation.

4. Find the product of the four following factors,

$$(a + b),\ (a^2 + a\,b + b^2),\ (a - b),\ (a^2 - a\,b + b^2).$$

What multiplier will render $\sqrt{a} - \sqrt{b}$ rational? Divide $a^{\frac{1}{3}}$ by $a^{\frac{1}{4}}$; and $\frac{a\,c - a\,d}{2\,b}\sqrt{a^2\,x - a\,x^2}$ by $\frac{a}{2\,b}\sqrt{a - x}$.

5. The sum of the progression of uneven numbers, 1, 3, 5, . . . continued to n terms, is equal to n^2. Prove this, and solve the equations:

$$(\alpha)\quad \frac{x + 5}{12} - \left(6\,x - 17 - \frac{7\,x - 40}{9}\right) = x - 30.$$

$$(\beta)\quad \left.\begin{aligned} x + y - z &= 8 \\ x + z - y &= 9 \\ y + z - x &= 10 \end{aligned}\right\}. \qquad (\gamma)\quad \left.\begin{aligned} x^2 + y^2 &= m \\ x + y &= n \end{aligned}\right\}.$$

Show that in every equation of the form $x^2 - a\,x + b = 0$, the two values of x are such that their sum is equal to a, and their product equal to b.

6. Define the sine and cosine of an angle, and trace their variations through four right angles. When the sines and cosines of two angles are given, show how to find the sine and cosine of their sum and difference.

7. Find the expression for the area of a triangle in terms of its sides.

(*Oct. 24th.*—Rev. Prof. HEAVISIDE.)

8. Find the equation to a straight line cutting its axes of rectangular co-ordinates in two given points. Find the rectangular equation to the ellipse, the centre being the origin of co-ordinates.

1850. *Monday, Oct. 28th.*—Examiner,—Mr. JERRARD.

1. Explain the method of extracting the square root of a number. What is the square root of 2, to five places of decimals?

2. A cistern is filled in 20 minutes by 3 pipes, one of which conveys 10 gallons more, and the other 5 gallons less, than the third per minute. The cistern holds 820 gallons. How much flows through each pipe in a minute?

3. Multiply together $1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4}$, and $1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4}$; and divide $a^n - b^n$ by $a - b$. What meaning do you assign to a^n when n is negative or fractional?

Reduce $\frac{x + y}{y} - \frac{2x}{x + y} + \frac{x^3 - x^2 y}{y^3 - x^2 y}$ to its most simple equivalent form, and discuss the case when $x = 1$, $y = 1$.

4. (1) The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical by 18; find the numbers. (2) Also solve the equations:

$$(\alpha)\quad \frac{9x + 7}{2} - \left(x - \frac{x - 2}{7}\right) = 36.$$

$$(\beta)\quad x^2 + 20x - 341 = 0.$$

$$(\gamma)\quad \left.\begin{array}{l} x^2 + y^2 = 650 \\ x + y = 36 \end{array}\right\}. \qquad (\delta)\quad \left.\begin{array}{l} a^x b^y = c \\ a_1^x b_1^y = c_1 \end{array}\right\}.$$

(3) If x_1, x_2, are the roots of the equation $a x^2 + b x + c = 0$, prove that

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{b^2 - 2ac}{ac}.$$

5. From a company of 50 men four are chosen every night to guard. On how many different nights can a different guard be posted; and on how many of these will any particular man be engaged?

6. Define the sine and cosine of an angle, and trace their variations through four right angles. Also prove that in a triangle the sides are proportional to the sines of the opposite angles.

7. Express the area of a triangle in terms of the sides. Take as an example the triangle of which the sides are 3, 4, and 5 units respectively.

(*Oct.* 30*th.*—Rev. Prof. HEAVISIDE.)

8. Find the equation to the straight line perpendicular to the line whose equation is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Find the rectangular equation to the parabola. Define the latus rectum, and find it in terms of the distance of the focus from the vertex.

1851. *Monday, Oct.* 27*th.*—Examiner,—G. B. JERRARD, Esq.

1. What is the interest of £273 15*s.* for a year, at $3\frac{1}{4}$ per cent.

2. Explain the rule of signs in the multiplication of algebraic quantities; and multiply

[1] $a^3 + a^2 b + a b^2 + b^3$ by $a - b$.
[2] $a^2 + a b + b^2$ by $a^2 - a b + b^2$.

Under what circumstances will $x^2 + a x + b$ be divisible by $x + y$?

3. Insert n arithmetical means between a and b, and find the sum of the series $1 + \frac{1}{2} + \frac{1}{4} +$ *ad infinitum*. Also solve the equations:

$$(\alpha)\ x - \frac{2}{3} - \frac{2x - 1}{7} = \frac{4}{21}. \qquad (\beta)\ \left.\begin{aligned} 2x + 3y &= 40 \\ 5x - 12y &= 61 \end{aligned}\right\}.$$

$$(\gamma)\ \left.\begin{aligned} x + y &= 18 \\ x^2 + y^2 &= 290 \end{aligned}\right\}. \qquad (\delta)\ \left.\begin{aligned} x + y &= a \\ x^3 + y^3 &= b \end{aligned}\right\}.$$

How many different arrangements can be formed of the letters in the word *engine?*

4. Define the term *logarithm*. What is meant by the *characteristic* and the *mantissa?* Explain the advantage of choosing 10 as a base of a system of logarithms. Given

Log. 3 = ·4771213, log. 7 = ·8450980,

to find log. 21, log. 210, log. 2·1.

5. Point out the use of the signs + and —, to indicate the directions of lines; and define the sine and cosine of an angle, tracing their variations through four right angles.

6. Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.

7. Express the area of a triangle in terms of its sides.

(*October* 29*th*.—Rev. Prof. HEAVISIDE.)

8. Define a conic section, distinguishing the several cases. Is the circle included in your definition of an ellipse? Find the rectangular equation to the ellipse, the centre being the origin.

1852. *Monday, Oct.* 25*th*.—Examiner,—G. B. JERRARD, Esq.

1. Convert $\frac{1}{7}$ into a recurring decimal; and show that the recurring periods for $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \ldots \frac{6}{7}$ will consist of the same digits. What is the form of those fractions which are convertible into finite decimals?

2. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour faster than the other, arrived there an hour sooner; required their rates of marching.

3. Under what circumstances will $x^n + y^n$ be divisible by $x + y$? Is $x^n + y^n$ ever divisible by $x - y$? Reduce

$$\frac{1}{(x+1)^3} - \frac{3}{2(x+1)^2} + \frac{5}{4(x+1)} - \frac{5}{4(x+3)}$$

to its most simple equivalent form.

4. Solve the equations:

$$(\alpha)\ \frac{5x+4}{17} - \left(2 - \frac{x}{2}\right) = 1 + \frac{x}{3}. \qquad (\beta)\ \left.\begin{aligned} 2x + 5y &= 41 \\ 5x - 2y &= 1 \end{aligned}\right\}.$$

$$(\gamma)\ \left.\begin{aligned} x^2 + y^2 &= 170 \\ xy &= 77 \end{aligned}\right\}. \qquad (\delta)\ \left.\begin{aligned} \frac{x+y}{x-y} &= \frac{a}{b} \\ x^2 - y^2 &= c \end{aligned}\right\}.$$

It is required to find four numbers in arithmetical progression, such that if they are increased by 2, 4, 8, and 15 respectively, the sums shall be in geometrical progression.

5. Show that the number of permutations of n things taken r together is $n(n-1)(n-2)\ldots(n-r+1)$.

In how many ways may seven balls be arranged in two divisions, so that the first division may contain three of the balls, the second four?

6. When the sines and cosines of two angles are given, show how to find the sine and cosine of their sum or difference; and thence deduce,

$$\text{Sin. } A + \text{sin. } B = 2 \text{ sin. } \frac{A+B}{2} \text{ cos. } \frac{A-B}{2},$$

$$\text{Sin. } A - \text{sin. } B = 2 \text{ cos. } \frac{A+B}{2} \text{ sin. } \frac{A-B}{2}.$$

7. To express the area of a triangle in terms of its sides.

(*Oct. 27th.*—Rev. Prof. Heaviside.)

8. Find the equation to a straight line passing through a given point, and perpendicular to a given straight line. Find the rectangular equation to the hyperbola, making either the centre or the vertex the origin.

1853. *Tuesday, October 25th.*—Examiner,—Mr. Jerrard.

1. Explain the difference between interest and discount. At what rate per cent., simple interest, will £225 amount to £256 10*s.* in 4 years?

Define stock. How much stock can be purchased by the transfer of £2,000 stock from the 3 per cents. at 90 to the 3½ per cents. at 96; and what change will be effected in income by it?

2. A ship having a crew of 26 persons carries provisions for 21 days; after having been at sea for 11 days, they pick up a party

from a wreck, and it is then found that the provisions will be exhausted in the course of five days; find the number of persons taken from the wreck.

3. What is meant by the base of a system of logarithms? Show that, in all systems, the logarithm of the base is 1, and the logarithm of 1 is 0. Point out the advantage of choosing 10 as the base of a system. Given log. 2 = ·3010300, log. 3 = ·4771213, to find log. 360, log. 36, log. 3·6.

4. Find the product of the four binomial factors, $x + a, x + b, x + c, x + d$; and thence deduce the expression when $a = b = c = d$. What multiplier will render $\sqrt{m} - \sqrt{n}$ rational? Divide $a^4 + a^2 b^2 + b^4$ by $a^2 + a b + b^2$; and $a^{\frac{1}{2}}$ by $a^{-\frac{1}{2}}$. Explain the meaning of fractional and negative indices.

5. Solve the equations:

$$(\alpha)\ 5\,x - \frac{2\,x - 18}{3} = 17 - (x - 53). \qquad (\beta)\ \left.\begin{aligned} 7\,x + 11\,y &= 154 \\ 5\,x - 3\,y &= 34 \end{aligned}\right\}.$$

$$(\gamma)\ \left.\begin{aligned} x^2 + y^2 &= m \\ x - y &= n \end{aligned}\right\}.$$

6. Assuming that

$$\text{sin. } (A + B) = \text{sin. } A \text{ cos. } B + \text{cos. } A \text{ sin. } B,$$

show how, from the mode of denoting the directions of lines by their signs, to deduce the expressions for sin. $(A - B)$, cos. $(A + B)$, cos. $(A - B)$. What is the numerical value of sin. 30°?

7. Prove that the sides of a triangle are proportional to the sines of the opposite angles.

(*Oct. 27th.*—Rev. Prof. Heaviside.)

8. Find by means of rectangular co-ordinates, the length of the perpendicular let fall from a given point on a given straight line. Define an ellipse, and find the rectangular equation to a given ellipse.

1854. *Monday, Oct. 23rd.*—Examiner,—G. B. Jerrard, Esq.

1. In what time will £350 amount to £402 10*s*. at 3 per cent., simple interest? A person invests £3400 in the 3 per cent. Consols at 95: what amount of stock does he receive, the brokerage being 2*s*. 6*d*. per cent.?

2. The quick-time or step, in marching, being two paces per second, and the length of each pace 28 inches; then at what rate per hour does a troop march on a route, and how long will they be in arriving at a garrison 20 miles distant, allowing a halt of one hour by the way to refresh?

3. Explain the rule of signs in the multiplication of algebraic quantities. What meaning do you assign to a^n, when n is negative or fractional? Under what circumstances will $x^n + y^n$ be divisible by $x + y$? Is $x^n + y^n$ ever divisible by $x - y$?

4. Required the number of permutations of n things taken r and r together.

How many different arrangements can be formed of the letters in the word *infinity?*

5. What number is that, the third part of which exceeds the fourth part by 16? Given

$$x\,y\,(x^2 + y^2) = 3 \ldots\ldots\ldots\ldots A$$
$$x^2 y^2 (x^4 + y^4) = 7 \ldots\ldots\ldots\ldots B$$

to find the values of x and y by a quadratic equation. Also find three numbers in arithmetical progression, such that the sum of their squares shall be 93; and if the first be multiplied by 3, the second by 4, and the third by 5, the sum of the products shall be 66.

6. Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.

7. Express, in a form adapted to logarithmic computation, the area of a triangle, in terms of the sides.

(*Oct. 25th.*—Rev. Prof. Heaviside.)

8. Assuming the method of representing the locus of points by rectangular co-ordinates, show that the general equation of the first degree between (x) and (y) represents a straight line.

Define the hyperbola, and find a rectangular equation to the hyperbola.

1855. *Wednesday, October 24th.*—Examiner,—Mr. Jerrard.

1. Convert $\frac{1}{7}$ into a recurring decimal; and show that if we take the successive multiples of $\frac{1}{7}$, namely $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$ . . . . the recurring periods will consist of the same digits. Determine the form of those fractions which are convertible into finite decimals.

2. What is meant by the terms *Involution* and *Evolution?* Find the continued product of the three factors $x + a$, $x + b$, $x + c$. What will the expression thence arising become, when $a = b = c$? Show how to extract the square root of a compound algebraical quantity. Ex. $a^4 - 2\,a^3\,x + 3\,a^2\,x^2 - 2\,a\,x^3 + x^4$. Explain the rule of pointing in the extraction of the square root of a number.

3. Solve the equations:

$$(1)\ \frac{3x - 7}{2} - \frac{19}{93}(5x - 4) = \frac{2}{3}. \qquad (2)\ \left.\begin{aligned} 11x + 7y &= 170 \\ 5x - 3y &= 34 \end{aligned}\right\}.$$

$$(3)\ \left.\begin{aligned} x + y &= a \\ x^2 + m\,x\,y + y^2 &= b \end{aligned}\right\}. \qquad (4)\ \left.\begin{aligned} y\,z\,u &= a^3 \\ x\,z\,u &= b^3 \\ x\,y\,u &= c^3 \\ x\,y\,z &= d^3 \end{aligned}\right\}.$$

and show that if α and β be the two roots of a quadratic equation

$$x^2 + \mathrm{a}\,x + \mathrm{b} = 0,$$

then will

$$x^2 + \mathrm{a}\,x + \mathrm{b} = (x - \alpha)\, x - \beta).$$

4. How does it appear that by means of a table of logarithms, *multiplication* may be performed by addition, *division* by subtraction, *involution* by multiplication, and *evolution* by division. Point out the advantage of choosing 10 as the base of a system of logarithms.

5. Explain the meaning of the term angle in trigonometry. What is a negative angle? Trace the variations of sin. A, cos. A, in sign and magnitude as A increases from 0° to 360°.

6. Given two sides and the included angle (a, C, b), to solve the triangle.

(*Oct. 25th*,—Rev. Prof. HEAVISIDE.)

7. Find the general equation to the circle referred to rectangular co-ordinates: how is the equation modified when the origin is [1] in the circumference, [2] in the centre of the circle?

Find the rectangular equation to the ellipse, the centre being the origin: what are the axes of the ellipse whose equation is

$$a^2 x^2 + b^2 y^2 = c^4 ?$$

1856. *Wednesday, Oct. 29th.*—Examiner,—G. B. JERRARD, Esq.

1. A wall was to be built 700 yards long in 29 days. Now, after 12 men had been employed on it for 11 days, it was found that they had completed only 220 yards of the wall. It is required to determine how many men must be added to the former, that the whole number of them may just finish the wall in the time proposed, at the same rate of working.

2. Explain the meaning of fractional and of negative exponents. Also extract the square root of $4\,a^4 + 12\,a^3 x + 13\,a^2 x^2 + 6\,a\,x^3 + x^4$, and reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to quantities which shall have the common exponent $\frac{1}{6}$.

3. Solve the equations:

$$(\alpha)\quad \frac{x}{2} - \frac{x}{3} + \frac{x}{4} = 5. \qquad (\beta)\quad \left.\begin{aligned} 3\,x + 5\,y &= 60 \\ x - 2\,y &= 9 \end{aligned}\right\}.$$

$$(\gamma)\quad \left.\begin{aligned} x^2 + a\,x\,y + y^2 &= m \\ x + b\,y &= n \end{aligned}\right\}. \qquad (\delta)\quad \left.\begin{aligned} \left(\frac{x}{a}\right)^{\alpha} \cdot \left(\frac{y}{b}\right)^{\beta} &= c \\ \left(\frac{x}{b}\right)^{\beta} \cdot \left(\frac{y}{a}\right)^{\alpha} &= d \end{aligned}\right\}.$$

If in the first of the equations (γ) we take $a = 0$, what will the expressions for x and y become (1) when $b = 1$, (2) when $b = -1$?

Explain the result when $a = 2$, $b = 1$.

4. Insert m arithmetical means between two given numbers; and show that if in any arithmetical progression we insert the same number of arithmetical means between each term and the following one, the new series will also be an arithmetical progression. Does an analogous proposition exist for geometrical progressions?

5. Find the number of permutations of n things taken r together. How many distinct triliteral words can be formed of eight consonants and one vowel, the vowel being always the central letter?

6. Given the sines and cosines of two angles, to find the sine and cosine of their sum or difference.

7. In any plane triangle $A\ B\ C$, express sin. $\frac{A}{2}$, cos. $\frac{A}{2}$ in terms of the sides; and thence deduce the expression for the area.

(*Oct.* 30*th.*—Rev. Prof. HEAVISIDE.)

8. Every equation of the first degree between x and y, is the equation to a straight line. Find the rectangular equation to a straight line passing through a given point, and making a given angle with a given straight line.

Find the rectangular equation to the parabola.

1857. *Wednesday, Oct.* 28*th.*—Examiner,—Mr. JERRARD.

1. If 180 men, in 6 days, working during 10 hours each day, can dig a trench 200 yards long, 3 wide, and 2 deep; in how many days, working during 8 hours each day, will 100 men dig a trench 360 yards long, 4 wide, and 3 deep?

2. Show how to find the highest common divisor of two algebraical expressions. Take, for example,

$$x^3 - b^2 x, \quad x^2 + 2\,b\,x + b^2.$$

3. Solve the equations:

$$(1) \qquad \frac{2\,x - 5}{3} - \tfrac{1}{2}(7\,x - 9) = x - 1.$$

$$(2) \qquad \left.\begin{aligned} 3\,x + 5\,y &= 34 \\ 17\,x - 7\,y &= 16 \end{aligned}\right\}.$$

and (3) divide the number 60 into two such parts that their product shall be to the sum of their squares in the ratio of 2 to 5.

4. If $a : b :: c : d$,

prove that $m\,a \pm n\,b : p\,a \pm q\,b :: m\,c \pm n\,d : p\,c \pm q\,d$.

How does it appear that Euclid's definition of proportion and the algebraical follow each from the other?

5. Define the sine and cosine of an angle; and trace the variations of sin. A, cos. A, in sign and magnitude as A increases from 0° to 360°.

6. Given two angles and the side between them ($A\ C\ b$), to solve the triangle.

(*Oct.* 29*th.*—Rev. Prof. HEAVISIDE.)

7. A straight line cuts the axes of co-ordinates at given distances from the origin, find the equation to the straight line, in terms of those distances.

If $y = 2x + 3$ be the equation to a straight line, find the equation to the line perpendicular to it from the origin: find also the length of this perpendicular.

Find a rectangular equation to the hyperbola.

1858. *Wednesday, Oct.* 27*th.*—Examiner,—G. B. JERRARD, Esq.

1. A railway train after travelling for one hour meets with an accident which delays it one hour, after which it proceeds at three-fifths of its former rate, and arrives at the terminus three hours behind time; had the accident occurred 50 miles further on, the train would have arrived 1 hour 20 minutes sooner. Required the length of the line.

2. Multiply together (1) $a + x$, $b + x$, and $c + x$; and resolve (2) $x^6 - a^6$ into four factors.

Explain the meaning of a^n when n is (3) fractional and (4) negative.

3. (1) Define the base of a system of logarithms. (2) What is meant by the characteristic and the mantissa? (3) Show that in the common system the characteristic of the logarithm of any number can be determined by inspection.

Given log. ·2 = ·301030, log. 3 = ·477121, find (4) log. 24, (5) log. 5·4, and (6) log. ·006.

4. Solve the equations:

$$(\alpha)\quad \frac{x+6}{2} - \frac{x-7}{3} = 2x - 13.$$

$$(\beta)\quad \left.\begin{array}{r} 7x + 11y = 57 \\ 13x - 21y = 23 \end{array}\right\}.$$

$$(\gamma)\quad \left.\begin{array}{r} x^2 + xy + y^2 = 7 \\ x - y = 3 \end{array}\right\}.$$

(1) What condition must be fulfilled in order that the two roots of the quadratic equation

$$ax^2 + bx + c = 0$$

may be equal? (2) How does it appear that a quadratic equation cannot have more than two roots?

5. Prove that

$$\sin.(A + B) = \sin. A \cos. B + \cos. A \sin. B;$$

and deduce formulæ for sin. $(A - B)$, cos. $(A + B)$, and cos. $(A - B)$.

6. To find an expression for the area of a triangle in terms of the sides, in a form adapted to logarithmic computation.

(*Oct.* 28*th.*—The Rev. J. W. L. Heaviside.)

7. What is the general form of the equation of the first degree between two variables? show that it represents a straight line in co-ordinate geometry.

Find the rectangular equation to a straight line in terms of the perpendicular upon it from the origin, and the angle which that perpendicular makes with one of the axes.

Find the general rectangular equation to a circle.

1859. *Tuesday, July* 19*th.*—Examiner,—E. J. Routh, Esq., M.A.

1. A mixture of black and green tea is sold so as to gain 4 per cent. on the outlay. If sold separately at the same price per lb., the gains would have been 5 and 3 per cent. respectively. In what proportion were the two kinds of tea mixed?

2. Simplify the expressions:

$$(1)\quad \frac{1}{13}\cdot\frac{\frac{2}{3}+\frac{4}{7}}{\frac{2}{3}-\frac{4}{7}}-\left(\frac{3}{4}+\frac{5}{6}-2\left(\frac{3}{8}-\frac{1}{12}\right)\right),$$

$$(2)\quad \frac{1}{\sqrt{1+x}+\sqrt{x}}+\frac{1}{\sqrt{1+x}-\sqrt{x}},$$

and (3) find the value of $x^2 + a\,x + b$ when

$$x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}.$$

3. Define the greatest common measure of two arithmetical, and also of two algebraical quantities. If H be the greatest common measure of two algebraical quantities P and Q, will it still be the greatest common measure when numerical quantities are given to the letters in H, P, and Q?

4. Give both the ordinary algebraic definition, and also Euclid's definition of proportion. Show how to deduce the former from the latter.

If $a : b :: c : d$, prove that

$$a\,d^3 + b\,c^3 : a^3\,d + b^3\,c :: c^2\,b^4 + a^2\,d^4 : (a\,b + c\,d)\,a\,b^3.$$

If b and d are very nearly equal, show that this ratio is very nearly equal to that of $3\,d - 2\,b : d$.

5. Show how to find the number of combinations of n things taken κ together.

The total number of combinations of $p + q$ things, of which p are of one sort and q of another, taking them first one, then two, three, &c., together, is $p + q + p\,q$.

6. Define (1) a geometrical progression. (2) Is it possible for three numbers to be both in arithmetical and geometrical progression?

(3) If s be the sum of a geometrical progression whose first term is a and last term l, and s' the sum of the reciprocals of the terms of the same series, then prove that $\frac{s}{s'} = a\,l$.

7. Investigate a rule to find the compound interest of any sum for n months at a given rate per cent. per annum.

(2) A person spends every year a certain fraction of his income, and continually adds the remainder to his capital; what fraction ought this to be, that, after a given number of years, his whole income may be increased n times?

8. Show that the equation

$$\frac{x + 6}{x - 1} + \frac{x - 6}{x + 1} = 2\,\frac{x^2 - 6}{x^2 - 1}$$

does not admit of any solution except $x = \infty$.

Solve

$$\left.\begin{aligned} 4\,x^2 + 7\,x\,y + 2\,y^2 &= 13 \\ 5\,x\,y + 7\,y^2 &= 12 \end{aligned}\right\}.$$

9. Define the characteristic of a logarithm. What are the characteristics of the logarithms of 234 and ·0067, the base of the system of logarithms being 7? What are the comparative advantages of the common and Napierian system of logarithms?

Find the value of $\overline{\cdot 001}|^{\cdot 001}$, having given

$$\text{log. } 9{\cdot}9328 = {\cdot}9960323,$$
$$\text{log. } 9{\cdot}9329 = {\cdot}9970367.$$

(*July* 19*th.*—Rev. Prof. HEAVISIDE.)

10. Define an ellipse; find a rectangular equation to the ellipse. If the equation is referred to the centre as origin, show how to refer it to the vertex; and if the equation is referred to the vertex as origin, show how to refer it to the centre.

11. How is an angle measured in trigonometry? What is the cosine of an angle? For what angle less than 90° is the sine equal to the cosine?

Express the numerical value (1) of a fifth part of two right angles, (2) of the complement of 62° 15′, (3) of the angle of a regular pentagon.

12. Prove $\tan. \overline{A + B} = \dfrac{\tan. A + \tan. B}{1 - \tan. A \,.\, \tan. B}$.

Account for the value of tan. $A + B$ given by this formula, if $A = B = 45^\circ$.

13. In every triangle the sines of the angles are proportional to the sides opposite to them.

Find the area of the triangle whose sides are 30, 40, 50 feet.

14. Given two angles and a side of a plane triangle; solve the triangle.

Given two sides and an angle opposite to one of them; show how to solve the triangle. Why is the solution in the latter case ambiguous?

Ex. Given $A = 52^\circ\ 15'\ 35''$, $a = 500$, $C = 88^\circ\ 30'$.
Find c, B.

$$\log.\ \sin.\ 88^\circ\ 30' = 9{\cdot}9998512.$$
$$\log.\ \sin.\ 52^\circ\ 15'\ 35'' = 9{\cdot}8980630.$$
$$\log.\ {}_{10}5 = {\cdot}6989700.$$
$$\log.\ {}_{10}6{\cdot}3206 = {\cdot}8007582.$$

How do the tabulated logarithms of the sines of angles differ from the logarithms of the sines calculated to a base 10?

1859. *Wednesday, Oct. 26th.*—Examiner,—Rev. J. W. L. HEAVISIDE.

1. A mass of lead-ore, weighing 800 grains Troy, was found to contain 6 grains of silver: what is the value of the silver in one ton of the ore, at the rate of *5s.* an ounce Troy, it being given that one pound avoirdupois contains 7000 grains Troy?

2. Prove

$$(1)\quad \frac{2\,x^3 - 13\,x + 15}{3\,x^3 + 9\,x^2 - 5\,x - 15} = \frac{2\,x^2 - 6\,x + 5}{3\,x^2 - 5}.$$

$$(2)\quad \frac{x^2 - x\,y + y^2}{x^2\,y^2}\left\{\frac{1}{x} + \frac{1}{y}\right\} - \frac{y^2 - y\,z + z^2}{y^2\,z^2}\left\{\frac{1}{y} + \frac{1}{z}\right\} = \frac{x^2 + x\,z + z^2}{x^2\,z^2}\left\{\frac{1}{x} - \frac{1}{z}\right\}.$$

3. Solve the following equations:

$$(1)\quad 3\,x - \frac{169 - 3\,x}{x} = 29. \qquad (2)\quad (x - a)\,(x - b) = (c - a)\,(c - b).$$

Find three numbers, in geometrical progression, the product of which is 729, and the sum of the squares 819.

4. Find the present value of an annuity, to be continued for any given number of years, at a given rate per cent., allowing compound interest. What does the expression for the present value become, when the annuity is perpetual? How many years' purchase must be paid for a freehold estate returning a given rent, allowing interest at five per cent.?

5. Explain why $\log._{10} 6{\cdot}25$, $\log._{10} {\cdot}000625$, $\log._{10} \frac{1}{16}$, have each the same mantissa; write down the logarithms of these numbers, it being given that $\log._{10} 2 = {\cdot}301030$. Account for the registered logarithms of the sines and cosines of angles being all positive. Given

$$\log. \tan. 46^\circ\, 17' = 10{\cdot}019462, \text{ find } \log. \cot. 46^\circ\, 17'.$$

6. Prove $\sin. (A + B) = \sin. A \cos. B + \cos. A \sin. B$, and deduce from it the expression for $\cos. (A - B)$. Could $\sin. (A + B)$ be expressed in terms of $\sin. A$ and $\sin. B$ only?

7. If two sides and the included angle of a triangle are given, show how to solve the triangle.

Ex. The two sides are 345, 174 feet respectively, and the included angle is 37° 20′; find the remaining angles of the triangle.

$$\log._{10} 5{\cdot}19 = {\cdot}715167, \quad \log. \tan. 71^\circ\, 20' = 10{\cdot}471298.$$
$$\log._{10} 1{\cdot}71 = {\cdot}232996, \quad \log. \tan. 44^\circ\, 17' = 9{\cdot}989127.$$

(*October* 27*th.*—E. J. Routh, Esq.)

8. If $y = a\,x + b$ be the equation to a straight line referred to rectangular co-ordinates, *state* the geometrical meanings of the quantities a and b.

Find the equation to the straight line which is drawn perpendicular to a given straight line from a given point.

State how a cone must be cut by a plane, that the section may be a parabola.

1860. *Tuesday, July* 17*th.*—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. If money invested in the 3 per cent. consols. give exactly 3 per cent., after the payment of one shilling in the pound income tax, find the price of the consols., allowing $\frac{1}{8}$th per cent. to the broker for the purchase.

A person invests £3,000 in the 3 per cents., at $94\frac{7}{8}$, and £2,000 in 6 per cent. Canada bonds, at $112\frac{1}{8}$; allowing in each case $\frac{1}{8}$th per cent. on the purchase to the broker, find the average per-centage obtained on the £5,000 invested.

2. Simplify the expression:

$$(1) \quad \left\{\frac{11\frac{3}{4} - 10\frac{1}{2}}{11\frac{3}{4} + 10\frac{1}{2}} \div \frac{10\frac{3}{8} + 11\frac{1}{8}}{10\frac{3}{8} - 9\frac{1}{8}}\right\} \times \frac{\frac{2}{7} + \frac{3}{11}}{\frac{2}{7} - \frac{3}{11}}.$$

(2) Divide ${\cdot}000024\dot{3}7\dot{4}$ by ${\cdot}00000\dot{2}43\dot{5}$, and (3) find the square roots of 9947716, and (4) ${\cdot}\dot{0}4938271\dot{6}$.

3. Divide

$$a^6 + a^5\,b + a^4\,b^2 + a^3\,b^2\,c + a^3\,b^3\,c + a^3\,b\,c^2 + a^3\,c^3 + a^2\,b^3\,c + a^2\,b\,c^3$$
$$+ a\,b^3\,c^2 + b^3\,c^3 + b\,c^5 \text{ by } a^3 + a^2\,b + b\,c^2;$$

and if n be a positive integer, prove that $x^{2n+1} + y^{2n+1}$ is always divisible by $x + y$.

Also simplify the expression:

$$\left\{\sqrt{\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}}+\sqrt{\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}}\right\}$$

$$\div\left\{\frac{\sqrt{3+2x}+\sqrt{3-2x}}{\sqrt{3+2x}-\sqrt{3-2x}}-\frac{\sqrt{3+2x}-\sqrt{3-2x}}{\sqrt{3+2x}+\sqrt{3-2x}}\right\}.$$

4. Define ratio and proportion; and deduce Euclid's definition V., Book V., from the algebraical definition. If $a : b = c : d = e : f$, prove that each of these ratios

$$\sqrt{\frac{a^2c^2}{e^2}+\frac{a^2e^2}{c^2}+\frac{c^2e^2}{a^2}} : \sqrt{\frac{b^2d^2}{f^2}+\frac{b^2f^2}{d^2}+\frac{d^2f^2}{b^2}}$$

$$= a^3df + c^3bf + e^3bd : b^3ce + d^3ae + f^3ac.$$

5. Find the number of permutations of n things r together, and deduce the number of combinations of n things r together.

If n represent the number of combinations of n things taken r together, prove by general reasoning that

$$(n)_r + (n)_{r-1} = (n+1)_r, \text{ and}$$

$$(m+n)_r = (m)_r + (m)_{r-1}(n)_1 + (m)_{r-2}(n)_2 + \ldots\ldots + (n)_r.$$

6. Find the sum of a series of quantities in geometrical progression, having given the first term and the common ratio.

If the sum of the nth and $(2n)$th terms of a geometrical progression be given, and also the sum of the $(2n)$th and $(3n)$th terms, find the first term and the common ratio.

Find the sum of n terms of the series of which the rth term is

$$2r + 3 + 2 \times 3^r.$$

7. (1) Explain what are meant by discount and present value; and (2) find expressions for the discount and present value of a sum due at the end of a given time, reckoning compound interest.

(3) A given sum is to be invested in an annuity such that each annual payment is one-third of the preceding, to continue for n years; reckoning compound interest at a given rate, find the payment for the first year.

8. Solve the equation:

$$6x^2 - 17ax + 4bx + 12a^2 - 7ab - 10b^2 = 0.$$

A number of men are first formed into a solid square, and afterwards into a hollow square three deep; the front presented in the latter formation has 75 men more than the front in the solid square: determine the number of men.

9. Define a logarithm, and find the logarithm of $a^{\frac{p}{q}}$ to the base $a^{\frac{m}{n}}$.

Show that the characteristic of the logarithm of any number or decimal can be determined by inspection when the base is 10; and prove the formula

$$\log._a N = \log._a 10 \cdot \log._{10} N.$$

Having given $\log._{10} 2 = \cdot 301030$

$\log._{10} 3 = \cdot 477121,$

find $\log._{10,000} \cdot 0000432$.

(July 17th.—Afternoon.—Trigonometry and Conics.)

10. Define the sine of an angle.

Prove that $\sin. A = \sin. (180^\circ - A)$; and discuss the case in which A is greater than two right angles.

Write down in one formula all the angles which have $\frac{1}{2}$ for their sine. Solve the equation

$$1 + \sin. 2\,\theta = \cos. \theta - \sin. \theta.$$

11. Prove that in any triangle

$$\cos. A = \frac{b^2 + c^2 - a^2}{2\,b\,c}.$$

Why is this formula not fit to determine the angles of a triangle when the three sides are given? What formula would you employ in such a case?

12. Prove that if A and B be any two angles,

$$\cos. (A - B) = \cos. A \,.\, \cos. B + \sin. A \,.\, \sin. B.$$

Prove also that

$$\sin. 2\,A \,.\, \sin. A = \cos. A - \cos. A \,.\, \cos. 2\,A;$$

and express $\cot. 2\,A$ in terms of $\cot. A$.

13. Find the equation to the perpendicular drawn from a given point on the straight line whose equation is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

What angles do the straight lines $x - \kappa\,y = 0$ and $y + \kappa\,x = 0$ make with each other?

14. Find the equation to the circle passing through the origin, and having its centre on the axis of x, and the radius of which is equal to a.

Interpret each of the equations

$$x^2 + y^2 = 0 \text{ and } x^2 - y^2 = 0.$$

A point moves so that the sum of the squares of its distances from the three angles of a triangle is constant. Prove that it moves along the circumference of a circle.

15. Investigate the equation to a parabola, $y^2 = 4\,a\,x$.

Explain the geometrical meaning of the constant a. Trace the form of the curve from its equation.

If $T A$, $T B$ be two tangents to a parabola cutting each other in the principal diameter, then if a third tangent cut them in P and Q, prove that $S P = S Q$ where S is the focus.

1860. *July 17th. First B. Sc. Examination.*
Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

☞ *The only separate Examination Paper for this degree. After this date, " the Papers and Examiners are the same as those on the same day at the First B.A. Examination."*

1. A contractor undertook to build a house in 21 days, and engaged 15 men to do the work. But after 10 days he found it necessary to engage 10 men more, and then he accomplished the work one day too soon. How many days behindhand would he have been if he had not engaged the 10 additional men?

2. Cube $(1 - x + 2 x^2)$, and determine the first six terms of the square of $1 + x + \frac{x^2}{1 . 2} + \frac{x^3}{1 . 2 . 3} + \frac{x^4}{1 . 2 . 3 . 4} + \ldots .$

Simplify $\frac{2 + \sqrt{3}}{1 + \sqrt{3}}$ and $\sqrt{\frac{\frac{-1}{1 - x^2} + 1}{\frac{1}{1 - \frac{1}{x^2}} - 1}}$

and extract the square root of $5 + \sqrt{6} + \sqrt{10} + \sqrt{15}$.

3. Solve the quadratic equation $a x^2 + b x + c = 0$, and show how to make the equation whose roots shall be α and β.

If α and β be the roots of the equation $a x^2 + b x + c = 0$, prove that $\frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{b (3 a c - b^2)}{c^3}$.

Solve the simultaneous equations

$$\left.\begin{aligned} 2 x^2 + 3 x y - 4 y^2 &= 10, \\ x^2 - 2 x y + 3 y^2 &= 3. \end{aligned}\right\}$$

4. Define a logarithm, and prove that

$$\log. (M N) = \log. M + \log. N.$$

What is the modulus of a system of logarithms? State the advantages of choosing 10 as the base of the system.

Explain generally how you would find, by the help of a table of proportional parts, the logarithm of a number consisting of six figures, from a table giving the logarithms of all numbers of five figures.

5. Define the sine and cosine of an angle. Find an expression for all the angles which have tan. α for their tangent.

Prove that if $A + B + C = 180$,

$$\frac{\sin. A + \sin. B - \sin. C}{\sin. A + \sin. B + \sin. C} = \tan. \frac{A}{2} . \tan. \frac{B}{2}.$$

6. In any triangle, prove that

(1) $\dfrac{\sin. A}{\sin. B} = \dfrac{a}{b}.$

(2) $\dfrac{\tan. \frac{A}{2}}{\tan. \frac{B}{2}} = \dfrac{s - b}{s - a}.$ Where $s = \dfrac{a + b + c}{2}.$

Explain how you would proceed to find the distance from a given point to any object on the other side of a river, supposing that you cannot cross the river, and that you have no instrument for measuring angles.

7. Find the equation to the straight line drawn from the given point (h, k) perpendicular to the given straight line $a x + b y + c = 0$.

Find also the equation to the system of circles passing through this given point, and touching this straight line. Prove, in any way, that the centres of these circles lie on a parabola.

8. A point P moves so that the sum of its distances from two given points, A and B, is constant. Find the locus of P.

Let S, H, be the foci of an ellipse, and A the extremity of the major axis. Then if P be any point on the ellipse, prove, in any way, that the bisectors of the angles $P S A$, $P H A$ meet in the tangent at P.

(July 17th.—Afternoon.)

9. Investigate the equation to the parabola, $y^2 = 4 a x$.

Explain the geometrical meaning of the constant a. Trace the form of the curve from its equation.

If $T A$, $T B$, be two tangents to a parabola, cutting each other in the principal diameter, then if a third tangent cut them in P and Q, prove that $S P = S Q$, where S is the focus.

1861. *July 16th.*—Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. A person invests in £10 railway shares when they are at a premium of ten shillings. At the end of a year he receives a guinea per share. What interest does he get?

If 81 bushels of wheat are consumed by 56 men in 5 days, how long will 16 men take to consume 28 bushels?

2. State and prove the rule for the determination of the fraction equivalent to a given recurring decimal.

Simplify $\dfrac{2\frac{1}{2} + 1\frac{1}{3}}{2\frac{1}{2} - 1\frac{1}{3}} \times \dfrac{1\frac{7}{12} - 1}{\frac{1}{4} + \frac{5}{3}}$; and divide ·000741 by 2·47.

3. Find the square root of $x^4 - 4x^3 + 2x^2 + 4x + 1$, and simplify

$$\sqrt{\frac{1 - x - \sqrt{2x + x^2}}{1 - x + \sqrt{2x + x^2}}} + \sqrt{\frac{1 - x + \sqrt{2x + x^2}}{1 - x - \sqrt{2x + x^2}}}.$$

4. If $\frac{a}{b}$ and $\frac{c}{d}$ be two equal fractions, prove that they are each $= \frac{a + c}{b + d}$.

If x vary as y, prove that $x^2 + y^2$ will vary as $x^2 - y^2$.

5. Explain the difference between combinations and permutations. Find the number of permutations of n things taken r together.

6. Find the sum of an arithmetical progression, the first and last terms and the number of terms being given.

Sum the following series to n terms—

$$2 + 3\tfrac{1}{2} + 5 + \ldots\ldots$$
$$2 + 3\tfrac{1}{2} + 6\tfrac{1}{8} + \ldots\ldots$$

Find the sum of n terms of the series whose nth term is $2(2^{n-1} + n) + 5$.

7. Solve the equations

$$2x^2 + 11x + 15 = 0.$$
$$bx^2 + 3ax + b - a = ax^2 + 3bx + a - b.$$
$$\left.\begin{aligned} x^2 + y^2 &= a^2 \\ x + y &= b \end{aligned}\right\}.$$

8. Find a number of two digits, which is three times the sum of its digits, and such that the difference between the digits is 5.

9. Define a logarithm, and prove that

$$\log. xy = \log. x + \log. y.$$

In what respects is the ordinary system of logarithms to base 10 more convenient, and in what respects less convenient than the Napierian system?

Having given log. 2 = ·301030, and log. 3 = ·477121, find log. ·0144.

Find the characteristic of log. 3·2 to base 5.

(*July 16th.—Afternoon.—Trigonometry and Conics.*)

10. Define the tangent of an angle, and prove that

$$\tan. A = -\tan. (180^\circ - A).$$

Find an expression for all the angles which have the same tangent as a given angle A, and solve the equation

$$3 \tan.^4 \theta - 10 \tan.^2 \theta + 3 = 0.$$

11. If A and B be each less than 90°, prove that

$$\sin. (A - B) = \sin. A \cos. B - \cos. A \sin. B;$$

also prove the formula

$$\text{cot. } A - 2 \text{ cot. } 2\,A = \text{tan. } A,$$

$$\frac{\sin.\left(\frac{\pi}{3} + \theta\right) + \cos.\left(\frac{5}{6}\,\pi - \theta\right)}{\sin.\left(\frac{5}{6}\,\pi - \theta\right) + \cos.\left(\frac{\pi}{3} + \theta\right)} = \text{tan. } \theta.$$

12. If two sides and the included angle of a triangle be given, show how to solve the triangle.

The angle A of a triangle $A\ B\ C$ is 60°, and the side $A\ C$ is twice the side $A\ B$; find the angles B and C.

13. Explain what is meant by the locus of an equation in x and y when x and y are the co-ordinates of a point referred to fixed axes.

Find the loci of the equations

(1) $x = 3\,y$. (2) $(x^2 - a^2)^2\,(x^2 - b^2)^2 + c^4\,(y^2 - a^2)^2 = 0$.

Also define the equation to a curve, and find the equation to a straight line.

14. If α, β be the co-ordinates of the centre, and C the radius of a circle, find its equation, the axes being rectangular.

Find the conditions that the circle may cut off from the axes chords of which the lengths are respectively a and b.

15. Define an ellipse, and find its equation referred to the centre and axes.

If the axes of an ellipse be given in direction, and if the ratio of their lengths be also given, through how many points can the curve be drawn?

16. Trace the form of the curve $y^2 - x^2 = a^2$, and find the equation to the tangent at the point (x, y).

Find x and y when the tangent cuts off a given area (a^2) from the axes.

1862. *July 22nd.—Morning,* 10 *to* 1.

Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Find the amount of £1,000 placed at simple interest for 11¾ years at 3¼ per cent.

The sum of £9,040 16*s.* is placed in the 3½ per cents. at 94; find the income obtained, allowing on the stock purchased ⅛th per cent. to the broker, and $\frac{1}{20}$th per cent. for other expenses.

2. Define a recurring decimal, and show how to reduce recurring decimals to ordinary fractions.

Express as a fraction $\cdot 20\dot{0}12\dot{3}$, and express as a recurring decimal $\cdot 0\dot{1}\dot{2} \div \cdot 00\dot{1}3\dot{2}$.

3. Multiply together $x^3 - 3x^2 + 12x - 1$, $4x^3 - x + 1$, and $x^3 + x^2$.

Extract the square root of

$$x^6 - 6x^4 + 4x^3 + 9x^2 - 12x + 4;$$

and resolve into quadratic factors the expression

$$(5x^2 - 11x + 12)^2 - (4x^2 - 15x + 6)^2.$$

4. Define variation.

If $A \propto B$ when C is constant, and if $A \propto \frac{1}{C}$ when B is constant, prove that generally $A \propto \frac{B}{C}$.

If $x \propto y$, prove that $x^2 + y^2 \propto xy$. Also, if $x^3 + \frac{1}{y^3} \propto x^3 - \frac{1}{y^3}$, prove that xy is constant.

5. Having given the number of permutations, find the number of combinations, of n things r together.

In how many ways can 8 coins be placed in a row on a table, and in how many of these will a particular coin be at an end?

How many different parcels of 4 can be made of these coins, and in how many of these will a particular coin occur?

6. Find the sum of n terms of a given geometric series, and also the sum to infinity when the common ratio is less than unity, explaining what is meant by the sum of an infinite series.

Sum the series

$$a - \frac{a^2}{r^2} + \frac{a^3}{r^4} - \ldots\ldots$$

to n terms and to infinity, a being less than r^2.

Also find the sum of n terms of the series of which the rth term is $2r + 2^r(1 + 2^r + 4^r)$.

7. Solve the equations

$$\overline{x+b}\,\overline{x+c} + \overline{x+c}\,\overline{x+a} = \overline{2x+a}\,\overline{x+b};$$

$$\left\{\begin{array}{l} x + y = a \\ ax + by = ab \end{array}\right\}; \qquad \left\{\begin{array}{l} x^2 + 4y^2 = 116 \\ xy = 20 \end{array}\right\};$$

$$x^4 + x^2 - 4\sqrt{x^4 + x^2 - 25} = 550.$$

8. A certain number has two digits, the sum of the squares of which is 130; and if the order of the digits be changed, the number is increased by 18. Find the number.

9. Define a logarithm, and find that of 81 to the base $\sqrt{3}$, and of x to the base $\frac{1}{\sqrt{x}}$.

Prove that $\log._a \frac{x}{y} = \log._a x - \log._a y$, and that $\log._a x = \log._a b \, . \, \log._b x$.

Having given $\log._{10} 2 = \cdot 301030$, find $\log._{10} 32$ and $\log._{100} 32$.

10. Obtain an algebraic expression for the simple interest of £P for n years, taking r as the interest of £1 for a year; and also for the present value, reckoning simple interest, of an annuity of £P to commence n years hence and to continue for n years.

(*July* 22.—*Afternoon*, 3 *to* 6.)

11. Define the complement of an angle. Prove that the sine of any angle is the cosine of its complement; and discuss the case in which the angle is greater than a right angle.

Find all the values of x which satisfy the equations:

$$(1)\ \sin. 2\,x = \cos. 3\,x;$$
$$(2)\ 1 + \sin. x = 2 \cos.^2 x.$$

12. Prove that when A lies between 90° and 180°, and B between 0° and 90°,

$$\sin. (A - B) = \sin. A \cos. B - \cos. A \sin. B.$$

Establish the equalities:

$$\sin. 3\,A \sin. A = \sin.^2 2\,A - \sin.^2 A;$$
$$\frac{1 + \cos. 2\,A}{\sin. 2\,A} = \cot. A.$$

13. Show how to solve a triangle when two of its sides and the included angle are given.

Two sides of a triangle are 6 and 8 feet, and the area is 12 square feet: find the third side.

14. Show that the equation

$$x \cos. a + y \sin. a - p = 0$$

represents a straight line and a straight line only.

What is the geometrical meaning of the constants a and p?

Give diagrams of the loci of the equations

$$x^2 y = 0, \quad x^2 + y^2 = 0, \quad x - y = 4.$$

15. Show how to determine the position and magnitude of the curve represented by the equation

$$A\,x^2 + A\,y^2 + B\,x + C\,y + E = 0.$$

Find the equations to the two straight lines joining the origin to the points of intersection of the two curves

$$\left.\begin{array}{l} x^2 + y^2 = a^2, \\ y = b\,x + c \end{array}\right\}.$$

16. Find the equation to the tangent to the conic

$$A\,x^2 + B\,y^2 = C$$

at any point $(x, y_{,})$.

If the tangent at any point P cut the axes of the curve, produced if necessary, in T and T', and if C be the centre of the curve, prove that the area of the triangle $T\,C\,T'$ varies inversely as the area of the triangle $P\,C\,N$, where $P\,N$ is the ordinate of P.

1863. *Tuesday, July* 21.—*Morning*, 10 *to* 1.
Examiners,—W. H. Besant, Esq., M.A., and E. J. Routh, Esq., M.A.

1. Find the rate per cent. at which £1,000 must be laid out at simple interest to become £1,100 in 5 years.

The three per cents. being at 93, determine the interest obtained for money thus invested.

2. Define a decimal fraction; and taking ·237 as an example, show from your definition that $\cdot 237 = \frac{237}{1000}$.

Divide 3085·5 by ·00051; and reduce to a vulgar fraction the recurring decimal 2·3017017017 . . . . . .

3. Extract the square root of $x^4 - 6\,x^3 = 7\,x^2 + 6\,x + 1$; and simplify

$$\frac{a^8 - 2\,a^4\,x^4 + x^8}{a^6\,x^2 + a^2\,x^6} \div \frac{a^4 - x^4}{a^4 + x^4}.$$

Find also the factors common to the two expressions

$x^4 + 6\,x^3 - 8\,x^2 + 1$, and $x^6 + 7\,x^5 - 3\,x^4 - 3\,x - 2$.

4. When are four quantities said to be proportionals?

If $a : b :: c : d$, prove that

$$\frac{a^3 + 3\,a^2\,b + b^3}{c^3 + 3\,c^2\,d + d^3} = \frac{a^3 + b^3}{c^3 + d^3}.$$

Show how to deduce the algebraical definition of proportionals from that given by Euclid; and consider the case in which the quantities are incommensurable.

5. Find the number of permutations of n things taken r together.

How many words of four consonants and one vowel can be formed from seven consonants and three vowels, the vowel being always in the middle place?

6. Given the first and last terms of an arithmetical progression of n terms, find the series.

Sum the following series:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots\ldots \text{ to infinity;}$$

$$1^2 + 2^2 + 4^2 + 8^2 + \ldots\ldots\ldots \text{ to } n \text{ terms.}$$

7. Solve the equation:

$$\frac{1}{x - a} - \frac{1}{x + a} = \frac{1}{x - b} - \frac{1}{x + b}.$$

What is the meaning of your result when $a = b$?

Solve also:

$$\left.\begin{array}{l} x^2 - y^2 = a^2 \\ x - y = b \end{array}\right\},$$

$$\text{and } 2\,x\sqrt{x^2 + x - 1} = 2\,x^2 - 5\,x + 2.$$

8. Two partners, A and B, gained £17 by trade. A's money was in trade one year and a half, and he received for his principal and interest £39. B's money was in trade two years, and he began with £45. What money did A begin with?

9. Define the logarithm of a number to any base. Can any number be taken as the base?

Prove that log. $(M\,N) =$ log. $M +$ log. N.

Find the characteristic of log. ·0003 to base 8.

(July 21st.—Afternoon 3 to 6.—Trigonometry and Conics.)

10. Define the sine of an angle, and prove that

$$\sin.\ \theta = \sin.\ (\pi - \theta).$$

Trace the changes in sign of the expressions sin. 4 θ and sin. (sin. θ) as θ changes from 0 to π; and solve the equations

$$(1)\quad \sin.\ 3\,\theta = \sin.\ 4\,\theta;$$

$$(2)\quad 2\ \sin.\ (\sin.\ \theta) = 1.$$

11. If $A + B$ be less than 90°, prove that

$$\sin.\ \overline{A + B} = \sin.\ A\ .\ \cos.\ B + \cos.\ A\ .\ \sin.\ B;$$

and prove the formulæ

$$4 \sin.\ 3\,A\ .\ \sin.\ 5\,A\ .\ \sin.\ 7\,A = \sin.\ A + \sin.\ 5\,A + \sin.\ 9\,A - \sin.\ 15\,A;$$

$$\frac{\tan.\ \overline{\theta + a} + \tan.\ \overline{\theta - a}}{\cot.\ \overline{\theta + a} + \cot.\ \overline{\theta - a}} = \tan.\ \overline{\theta + a}\ .\ \tan.\ \overline{\theta - a}\ .$$

12. Find an expression for the cosine of an angle of a triangle in terms of the sides.

Having given the sides of a triangle, investigate a convenient formula for determining its angles by logarithmic computation.

If one of the angles be very small, will your formula determine the angle accurately? and if not, what modification would you employ?

13. Define the equation to a curve; and find the equation to a straight line in the form

$$\frac{x}{a} + \frac{y}{b} = 1.$$

Find the equation to a straight line which passes through a given point, and cuts off a given area from the co-ordinate axes, determining the condition that this may be possible.

14. Interpret the equations

$$(1)\quad \overline{x-a}|^2 + \overline{y-b}|^2 = c^2;$$
$$(2)\quad x^2 + y^2 + a^2 + b^2 = 2\,a\,x + 2\,b\,y.$$

Find the equation to the circle passing through the origin and the points $(a, b,)$ (b, a); and determine the lengths of the chords it cuts from the axes.

15. Define a parabola, and find its equation referred to its axis and the tangent at its vertex as co-ordinate axes.

If the tangent and normal at a point P of a parabola meet the tangent at the vertex (A) in K and L respectively, prove that

$$K\,L^2 :: S\,P^2 : S\,P - A\,S : A\,S,$$

S being the focus.

16. Find the equations to the tangent and normal at any point of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If ϕ be the eccentric angle of the point, prove that the equation to the normal is

$$\frac{a\,X}{\cos.\,\phi} - \frac{b\,Y}{\sin.\,\phi} = a^2 - b^2;$$

and hence find the greatest area cut off by the normal from the axes.

1864. *July* 19*th*, *Morn.* 10 *to* 1.—Examiners,—W. H. BESANT, Esq., M.A. and ISAAC TODHUNTER, Esq., M.A., F.R.S.

1. Distinguish between vulgar and decimal fractions, and define a recurring decimal.

Express as a decimal the product of $\frac{3}{5}$, $\frac{9}{8}$, and $\frac{14}{25}$; and divide ·137052 by ·0000324.

Also find the square root of $\cdot\dot{4}$, and of 10·4976.

2. If the 3 per cent. consols be at 91⅛, what sum of money must be expended in the purchase of stock in order to obtain an income of £528 a year?

If the purchaser afterwards sell out at 92¼, and invest the proceeds in mortgages at 5 per cent. per annum, what will be the increase of his income?

3. Prove that $b\,c\,(c-b) + c\,a\,(a-c) + a\,b\,(b-a) = (b-c)(c-a)(a-b)$; and divide $x^6 + 4\,x^5 - 3\,x^4 - 16\,x^3 + 2\,x^2 + x + 3$ by $x^3 + 4\,x^2 + 2\,x + 1$.

Also simplify the expression

$$\frac{x^6+y^6}{x^6-y^6} \times \frac{x-y}{x+y} \div \frac{x^4-x^2\,y^2+y^4}{x^4+x^2\,y^2+y^4},$$

and determine its value when $x = y$.

4. Give the algebraic definition of proportion; and deduce from it Euclid's test of proportion (Def. 5, Book V.).

If $a : b :: c : d$, prove that

$$ma + nb : ma - nb :: mc + nd : mc - nd;$$

and $a^4 + 4a^3b + b^4 : a^4 - 4a^3b + b^4 :: c^4 + 4c^3d + d^4 : c^4 - 4c^3d + d^4$.

5. Find the sum of n terms of a given geometric progression; and, if the common ratio be less than unity, find the sum of the series continued to infinity.

$$\text{Example. } \frac{4}{15} + \frac{1}{5} + \frac{3}{20} + \ldots\ldots \text{ to infinity.}$$

If r be the common ratio, s the sum of n terms, and σ the sum of the squares of the same n terms, prove that

$$\sigma (1 + r)(1 - r^n) = s^2 (1 - r)(1 + r^n).$$

6. Find the number of permutations of n things taken r together.

Four letters are written, and four envelopes directed. Determine (1) the total number of ways in which the letters may be put into the envelopes; (2) the number of ways in which the letters may all go wrong.

7. Define a logarithm; and find the logarithm of 81, (1) to the base 3; (2) to the base $\frac{1}{\sqrt{3}}$.

Prove that

$$\log._a \frac{M}{N} = \log._a M - \log._a N;$$

and having given $\log._{10} 2 = \cdot301030$, find $\log._{10} 25$, and $\log._{100} 25$.

8. A sum of £P is put out at simple interest for n years. Find an expression for its amount at the end of that time.

If £P be due n years hence, find its present value, reckoning simple interest: and, if the interest be $3\frac{1}{2}$ per cent. per annum, find what must be the value of n in order that the present value may be £$\frac{1}{2}P$.

9. Solve the equations

(α) $(x - a)(x - 2a) = (x - 3a)(x - 4a)$;

(β) $x^2 - 32x + 255 = 0$;

(γ) $x^2 + 2ax + 4a\sqrt{x^2 + 2ax} = 12a^2$;

(δ) $\left\{\begin{array}{r} x^2 + y^2 = 514 \\ xy = 255 \end{array}\right\}$.

10. A certain number, consisting of three digits, exceeds ten times the sum of its digits by 36. The third digit is equal to the sum of the first two; and the second digit increased by unity is equal to the product of the first and third digits. Find the number.

(*July* 19*th.—Afternoon,* 3 *to* 6.—*Trigonometry and Conics.*)

11. Define the cosine of an angle. Show that

$$\cos.\,(A - B) = \cos.\,A\,\cos.\,B + \sin.\,A\,\sin.\,B.$$

Find the cosine of 15°.

12. Show that in any triangle the sides are proportional to the sines of the opposite angles. Hence, deduce the expression for the cosine of an angle of a triangle in terms of the sides.

13. Find the equations to the straight lines which pass through a given point, and make a given angle with a given straight line.

Example. Find the equations to the lines which pass through the origin, and are inclined at an angle of 75° to the straight line $x + y + \sqrt{3}\,(y - x) = a$.

14. Investigate the equation to the tangent at any point of a parabola.

From an external point (h, k) two tangents are drawn to the parabola $y^2 = 4\,a\,x$. Find the area of the triangle formed by the tangents and the chord of contact.

15. Find the equation to the normal at any point of the ellipse.

Write down the equations to the normals at the ends of the latera recta.

16. Find the equation to the hyperbola, considering it as the locus of a point which moves, so that its distance from one fixed point differs by a constant quantity from its distance from another fixed point.

Show that the hyperbola has two asymptotes.

1865. *Tuesday, July 18th.—Morning.* Examiners,—
E. J. ROUTH, Esq., M.A., and ISAAC TODHUNTER, Esq., M.A.

1. What is the annual interest obtained if £770 14*s*. 7*d*. be invested in the purchase of 3 per cent. stock at $94\frac{3}{8}$?

2. Add together $\frac{5}{16}$, $\frac{7}{32}$, $\frac{3}{8}$, ·046875, and 1·23.

Simplify $\frac{\cdot 0075 \times 2\cdot 1}{\cdot 0175}$ and $\frac{4\cdot 255 \times \cdot 0064}{\cdot 00032}$.

3. Divide $(a^3 - 9 a^2 b + 23 a b^2 - 15 b^3)(a - 7 b)$ by $a^2 - 8 a b + 7 b^2$.

Find the highest common divisor of $12 x + 13 x^2 + 6 x + 1$ and $16 x^3 + 16 x^2 + 7 x + 1$.

4. Give the algebraical definition of proportion.

If $a : b :: c : d$ and $p : q :: r : s$, show that $\frac{a + c}{b + d} \cdot \frac{p - q}{r - s} = \frac{a - c}{b - d} \cdot \frac{p + q}{r + s}$.

5. Find the sum of a given number of terms of an arithmetical progression, the first term and the common difference being supposed known.

How many terms must be taken of the series 15, 12, 9 . . . that the sum may be 45?

Find the sum of 8 terms of the geometrical progression 4, 2, 1, $\frac{1}{2}$ . . . .

6. Find the number of combinations of n things taken r at a time.

Out of 12 consonants and 4 vowels how many words can be formed, each containing 3 consonants and 2 vowels?

7. Find the present value of an annuity to continue for a certain number of years, allowing compound interest.

If 20 years' purchase must be paid for an annuity to continue a certain number of years, and 24 years' purchase for an annuity to continue twice as long, find the rate per cent.

8. Define a logarithm; and show that in the common system of logarithms the characteristic of a logarithm can be determined by inspection.

Given $\log_{10} 2 = \cdot 301030$, find $\log_{10} \sqrt{\cdot 00025}$.

9. Solve the following equations:—

(1) $(x - 10)^3 (x - 6) = (x - 8)^3 (x - 12)$;

(2) $\begin{cases} 3 x^2 + 5 x - 8 y = 36, \\ 2 x^2 - 3 x - 4 y = 3; \end{cases}$

(3) $\frac{a + b}{x + b} - \frac{a + c}{x + c} = \frac{4 a}{x + a}$.

10. There is a certain rectangular floor, such that if it had been two feet longer and one foot narrower the area would have been the same; and if it had been four feet shorter and three feet broader, the area would also have been the same. Determine the length and breadth of the floor.

(*July* 18*th.—Afternoon.—Trigonometry and Conics.*)

11. Define the sine of an angle. Find all the angles whose sine is the same as sine α. Given the tangent of an angle, deduce an expression for the sine of the same angle.

Prove in a geometrical manner that

$$\sin.(A - B) = \sin. A \,.\, \cos. B - \cos. A \,.\, \sin. B;$$

where A has a value about 300 and B about 40 degrees.

12. Show how to find the angles of a triangle whose sides are given. Find also an expression for its area.

The diagonals of a quadrilateral field are a and b, and the acute angle between them is θ. Find the area of the field.

13. Find the equation to a straight line in the form $\frac{x}{a} + \frac{y}{b} = 1$.

Give a diagram showing the position of the lines $2x + 3y + 6 = 0$, and $x^2 - y^2 = 0$.

14. Find the equation to the tangent to the parabola $y^2 = 4ax$.

A straight line, $y = bx + c$, is drawn cutting the curve. Prove that the equation to the two straight lines drawn from the origin to the points of intersection are given by

$$cy^2 - 4axy + 4abx^2 = 0.$$

15. Assuming that the sum of the distances of any point of an ellipse from two fixed points, called the foci, is constant, find the equation to the curve in rectangular co-ordinates.

16. The equation to an hyperbola being $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, find the condition that its asymptotes may be at right angles.